ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Circuits and Systems I

LECTURE #11

Linearity, Time Invariance, and Convolution

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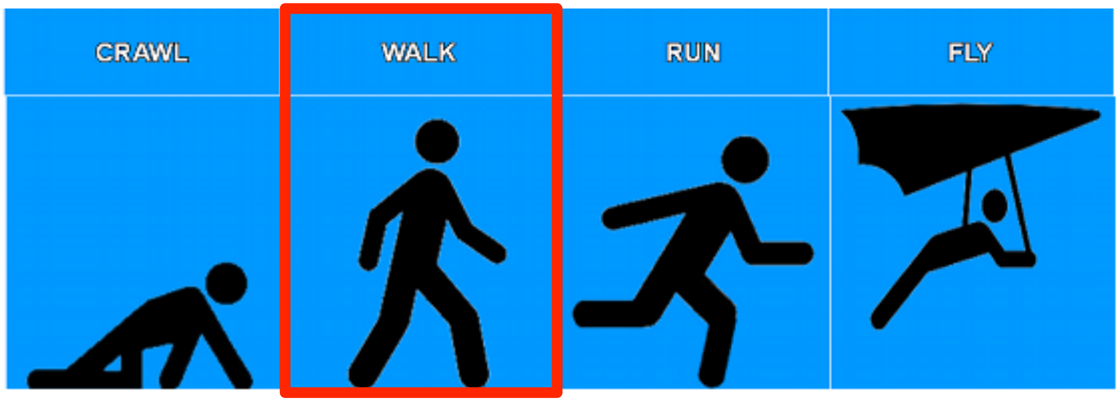
***CSI* Progress Level:**



Lecture Objectives

- GENERAL PROPERTIES of FILTERS
  - LINEARITY
  - TIME-INVARIANCE
  - ==> **CONVOLUTION**

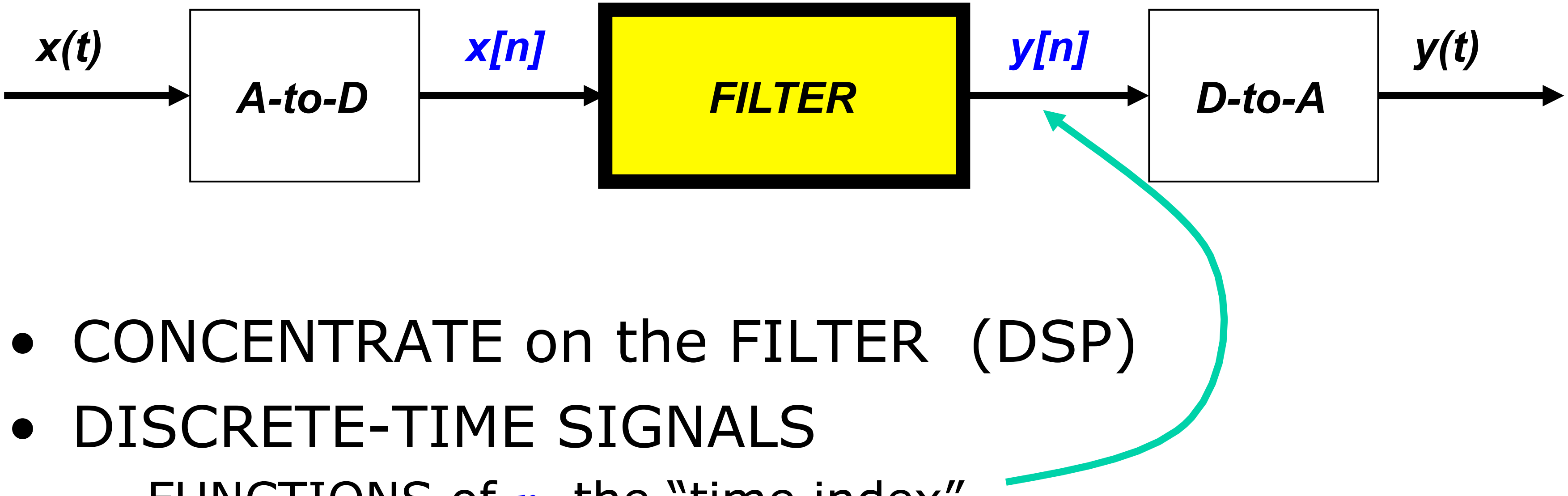
LTI SYSTEMS

- BLOCK DIAGRAM REPRESENTATION
  - Components for Hardware
  - Connect Simple Filters Together to Build More Complicated Systems

Overview

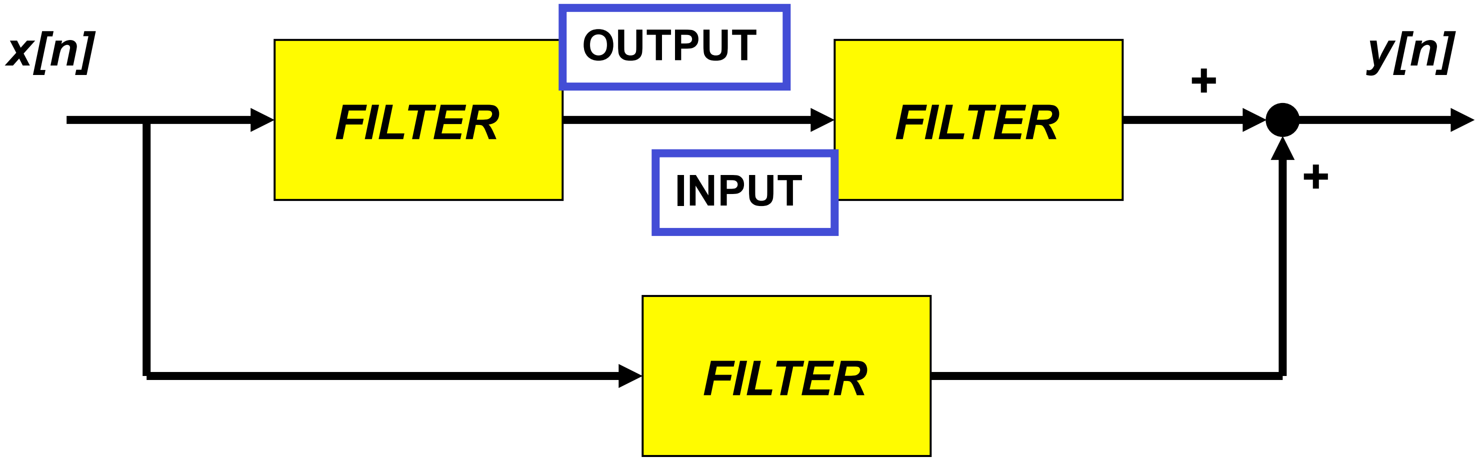
- IMPULSE RESPONSE, $h[n]$
  - FIR case: same as $\{b_k\}$
- CONVOLUTION
  - GENERAL: $y[n] = h[n] * x[n]$
  - GENERAL CLASS of SYSTEMS
  - LINEAR and TIME-INVARIANT
- ALL **LTI** systems have $h[n]$ & use convolution

Digital Filtering



- CONCENTRATE on the FILTER (DSP)
- DISCRETE-TIME SIGNALS
  - FUNCTIONS of n, the "time index"
  - INPUT $x[n]$
  - OUTPUT $y[n]$

Building Blocks



- BUILD UP COMPLICATED FILTERS
  - FROM SIMPLE MODULES
  - Ex: FILTER MODULE MIGHT BE 3-pt FIR

General FIR Filter

- FILTER COEFFICIENTS $\{b_k\}$
  - DEFINE THE FILTER

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

  - For example,

$$b_k = \{3, -1, 2, 1\}$$

$$y[n] = \sum_{k=0}^{3} b_k x[n-k]$$

$$= 3x[n] - x[n-1] + 2x[n-2] + x[n-3]$$

MATLAB for FIR Filter

- `yy = conv(bb,xx)`
  - VECTOR **bb** contains Filter Coefficients
  - DSP-First: `yy = firfilt(bb,xx)`
- FILTER COEFFICIENTS $\{b_k\}$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

conv2()
for images

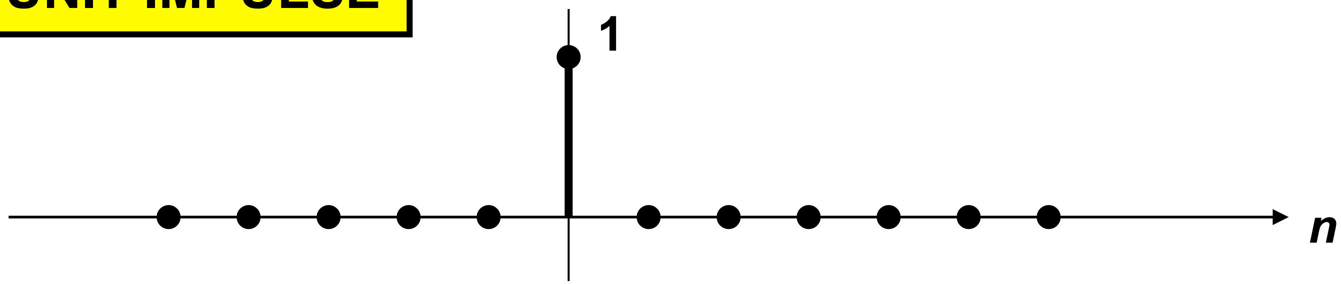
Special Input Signals

- x[n] = SINUSOID
- x[n] has only one NON-ZERO VALUE

FREQUENCY RESPONSE

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

UNIT-IMPULSE



FIR Impulse Response

- Convolution = Filter Definition
  - Filter Coeffs = Impulse Response

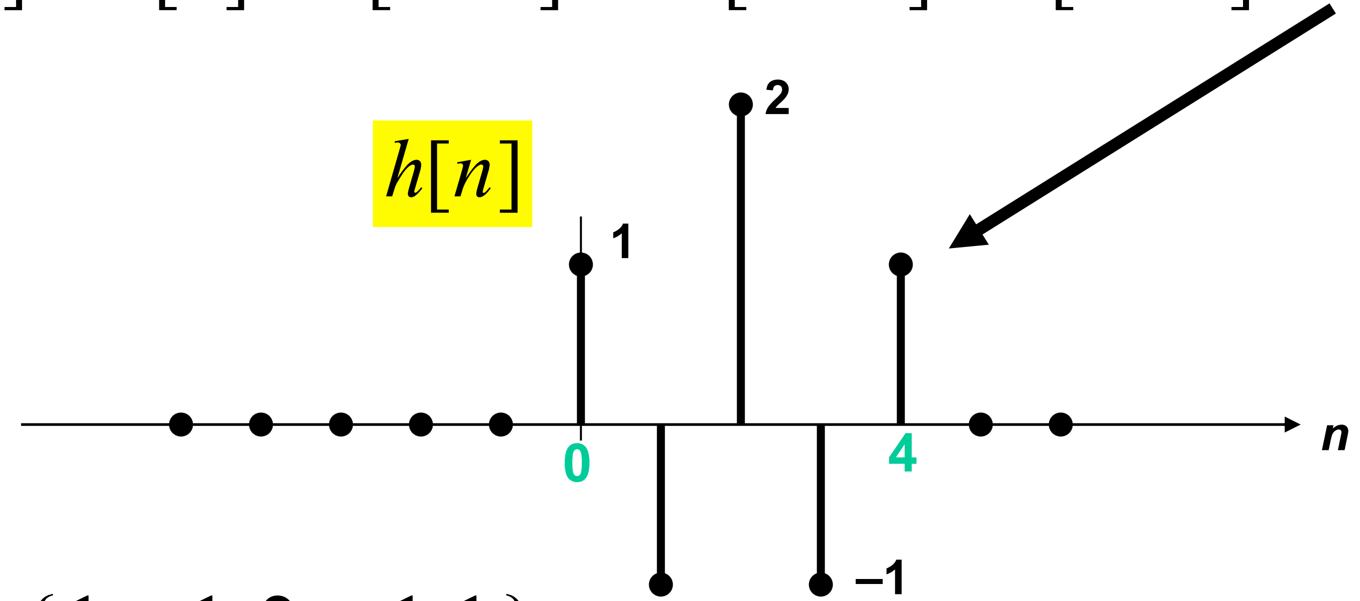
n	$n < 0$	0	1	2	3	$\ldots$	M	$M+1$	$n > M+1$
$x[n] = \delta[n]$	0	1	0	0	0	0	0	0	0
$y[n] = h[n]$	0	b_0	b_1	b_2	b_3	$\ldots$	b_M	0	0

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

Mathematical Formula for h[n]

- Use SHIFTED IMPULSES to write $\boldsymbol{h}[\boldsymbol{n}]$

$$h[n] = \delta[n] - \delta[n-1] + 2\delta[n-2] - \delta[n-3] + \delta[n-4]$$



$$b_k = \{\,1, -1, 2, -1, 1\,\}$$

LTI: Convolution Sum

- **Output = Convolution of x[n] & h[n]**
  - NOTATION:
  - Here is the FIR case:

$$y[n] = h[n] * x[n]$$

FINITE LIMITS

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

FINITE LIMITS

Same as $\mathbf{b_k}$

CONVOLUTION Example

$$h[n] = \delta[n] - \delta[n-1] + 2\delta[n-2] - \delta[n-3] + \delta[n-4]$$

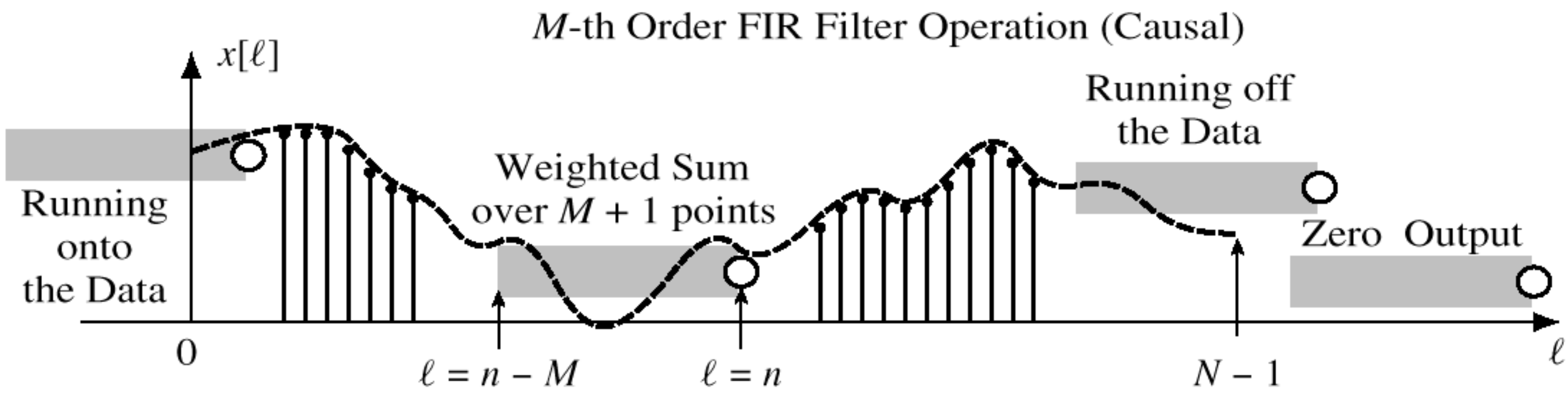
$$x[n] = u[n]$$

n	-1	0	1	2	3	4	5	6	7
$x[n]$	0	1	1	1	1	1	1	1	...
$h[n]$	0	1	-1	2	-1	1	0	0	0
$h[0]x[n]$	0	1	1	1	1	1	1	1	1
$h[1]x[n-1]$	0	0	-1	-1	-1	-1	-1	-1	-1
$h[2]x[n-2]$	0	0	0	2	2	2	2	2	2
$h[3]x[n-3]$	0	0	0	0	-1	-1	-1	-1	-1
$h[4]x[n-4]$	0	0	0	0	0	1	1	1	1
$y[n]$	0	1	0	2	1	2	2	2	...

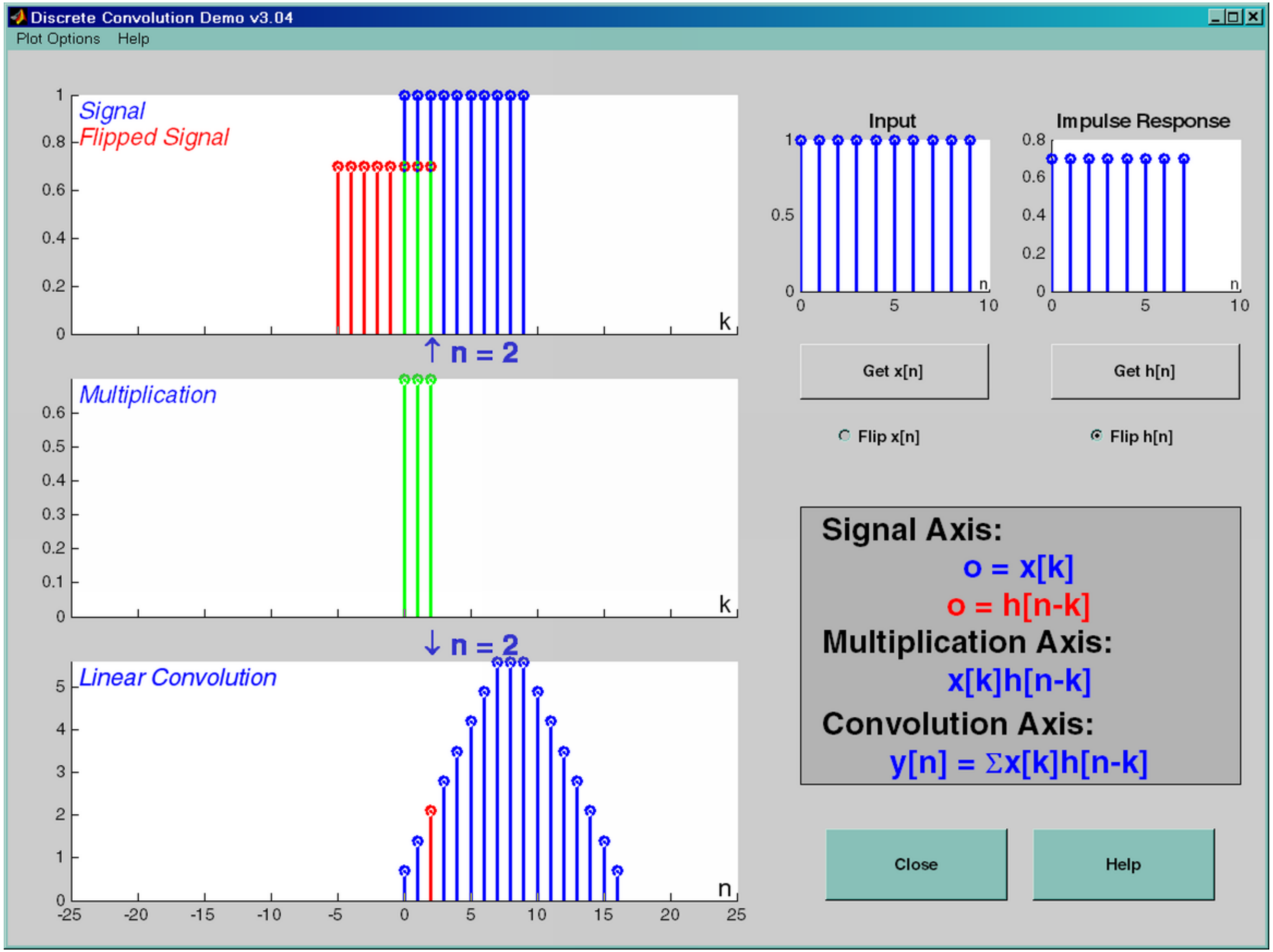
GENERAL FIR FILTER

- SLIDE a Length-L WINDOW over x[n]

$$y[n] = \sum_{k=0}^{M} b_k \, x[n-k]$$



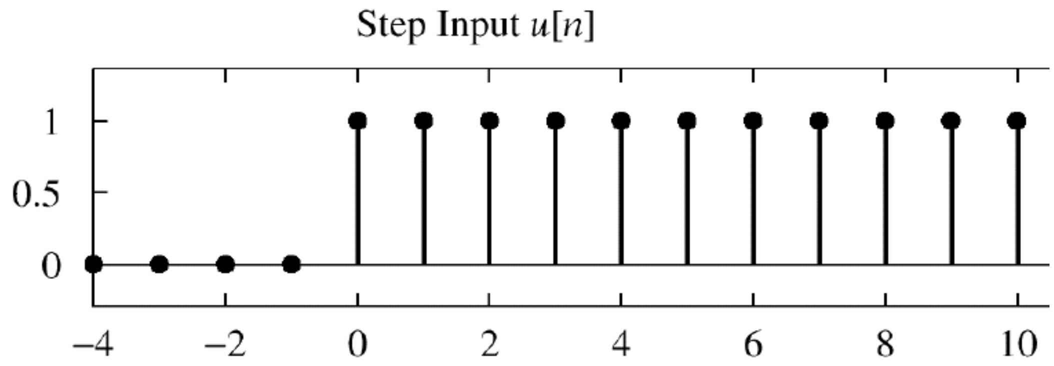
DCONVDEMO: MATLAB GUI



Pop Quiz

- FIR Filter is "FIRST DIFFERENCE"
  - $y[n] = x[n] - x[n-1]$
- INPUT is "UNIT STEP"

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



- Find $y[n]$

$$y[n] = u[n] - u[n-1] = \delta[n]$$

Hardware Structures

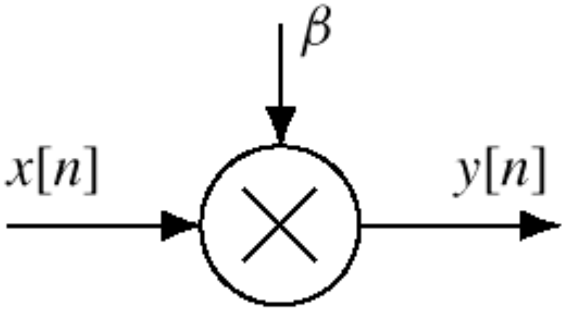
$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

- INTERNAL STRUCTURE of "FILTER"
  - WHAT COMPONENTS ARE NEEDED?
  - HOW DO WE "HOOK" THEM TOGETHER?
- SIGNAL FLOW GRAPH NOTATION

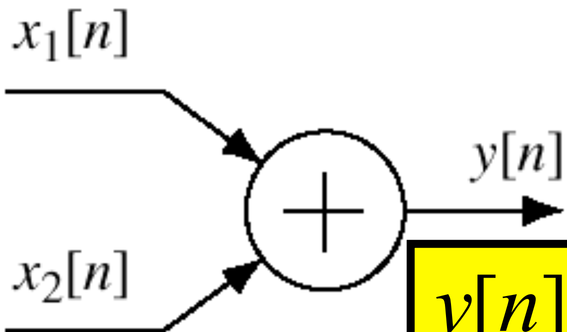
Hardware Atoms

- Add, Multiply & Store

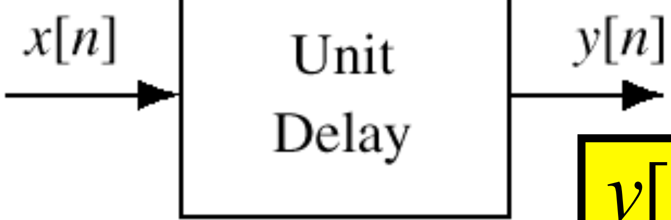
$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$



$$y[n] = \beta x[n]$$



$$y[n] = x_1[n] + x_2[n]$$



$$y[n] = x[n-1]$$

FIR Structure

- Direct Form

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

SIGNAL FLOW GRAPH

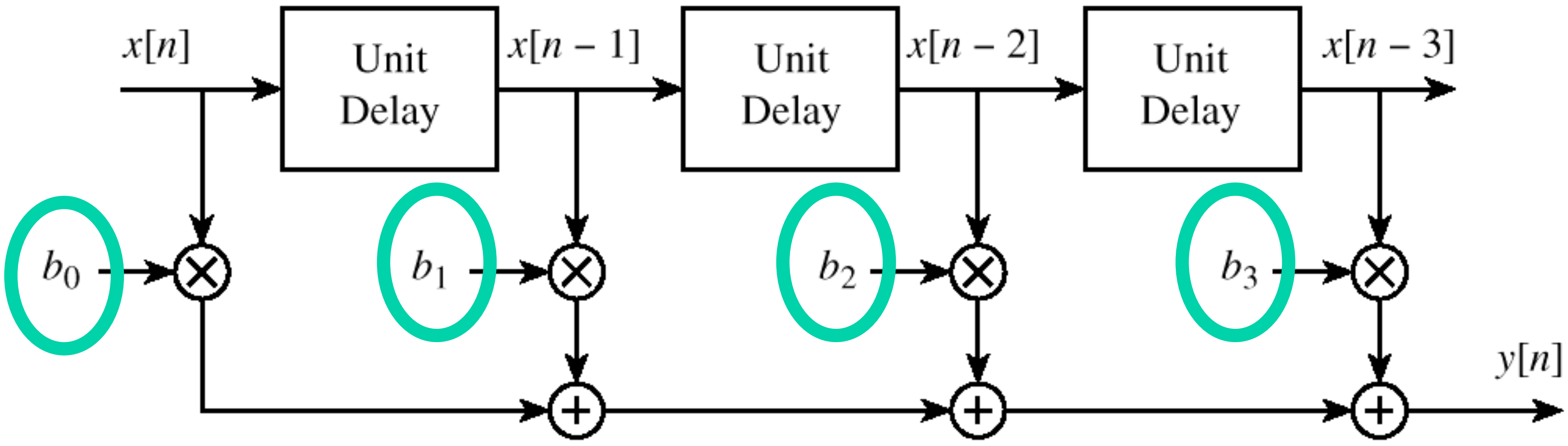
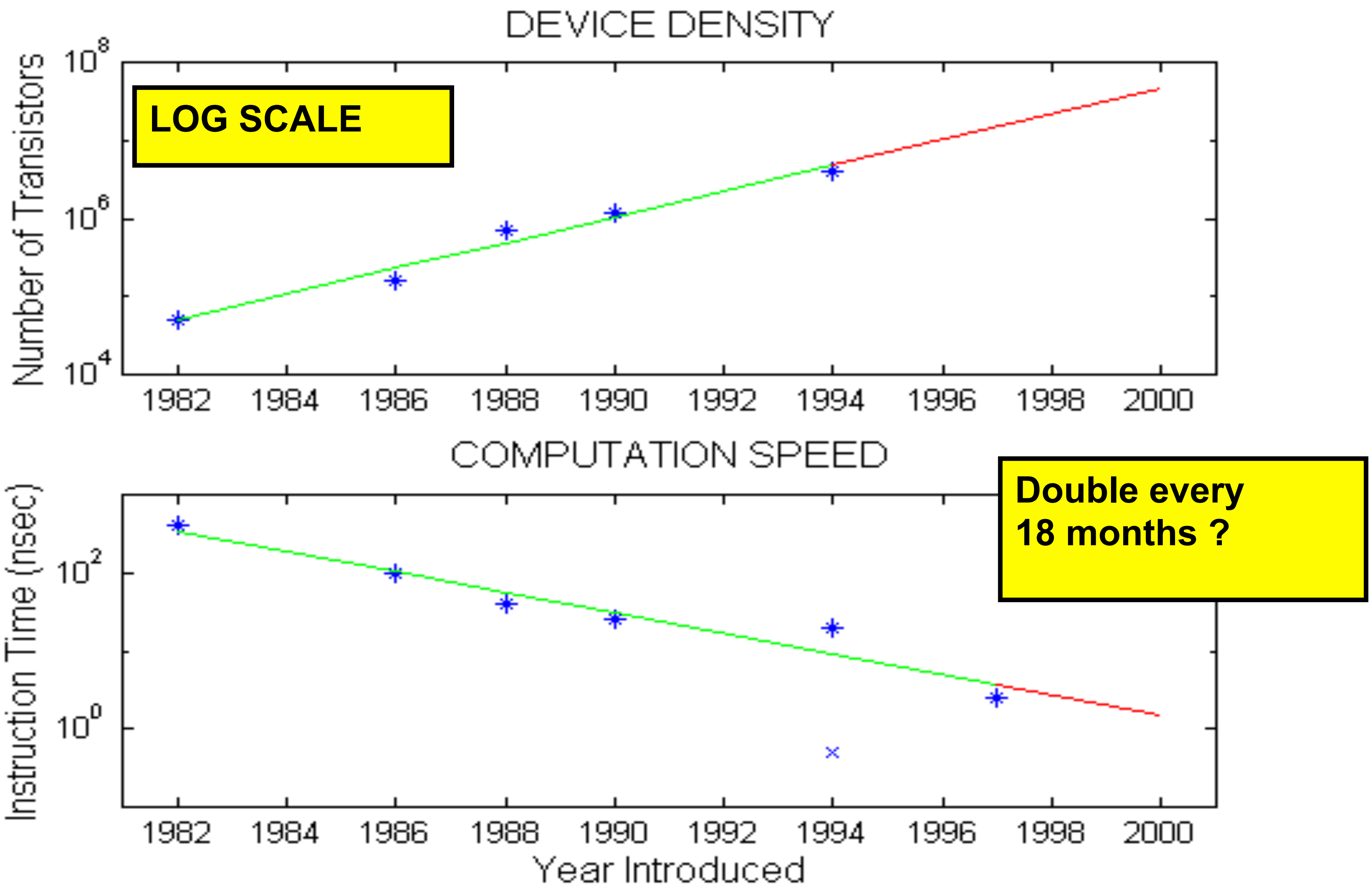


Figure 5.13 Block-diagram structure for the *M*th order FIR filter.

Moore's Law for TI DSPs



System Properties

- MATHEMATICAL DESCRIPTION
- TIME-INVARIANCE
- LINEARITY
- CAUSALITY
  - “No output prior to input”

Time-Invariance

- IDEA:
  - "Time-Shifting the input will cause the same time-shift in the output"
- EQUIVALENTLY,
  - We can prove that
    - The time origin (n=0) is picked arbitrary

TESTING Time-Invariance

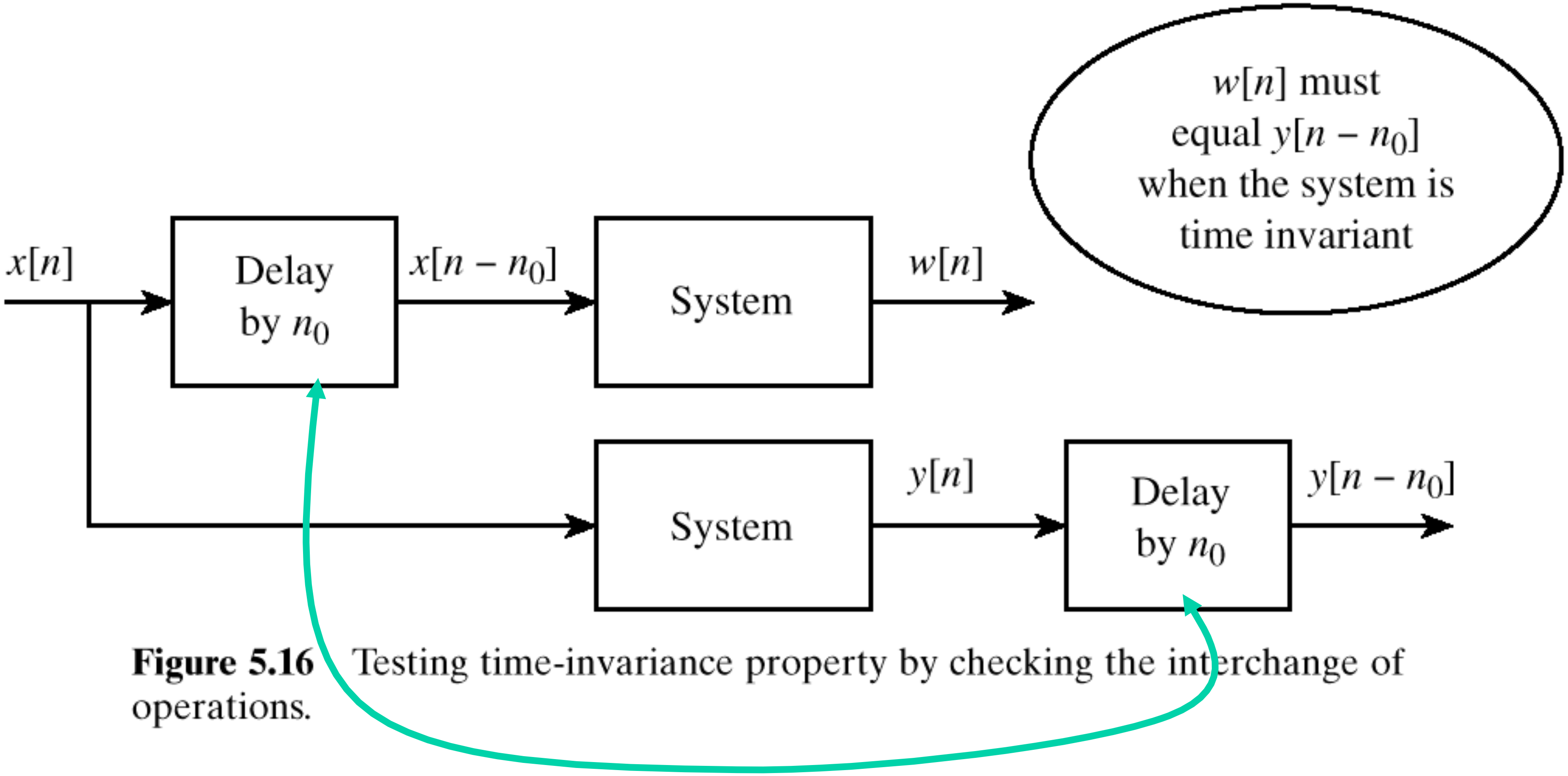


Figure 5.16 Testing time-invariance property by checking the interchange of operations.

Linear System

- LINEARITY = Two Properties
- SCALING
  - "Doubling x[n] will double y[n]"
- SUPERPOSITION:
  - "Adding two inputs gives an output that is the sum of the individual outputs"

TESTING Linearity

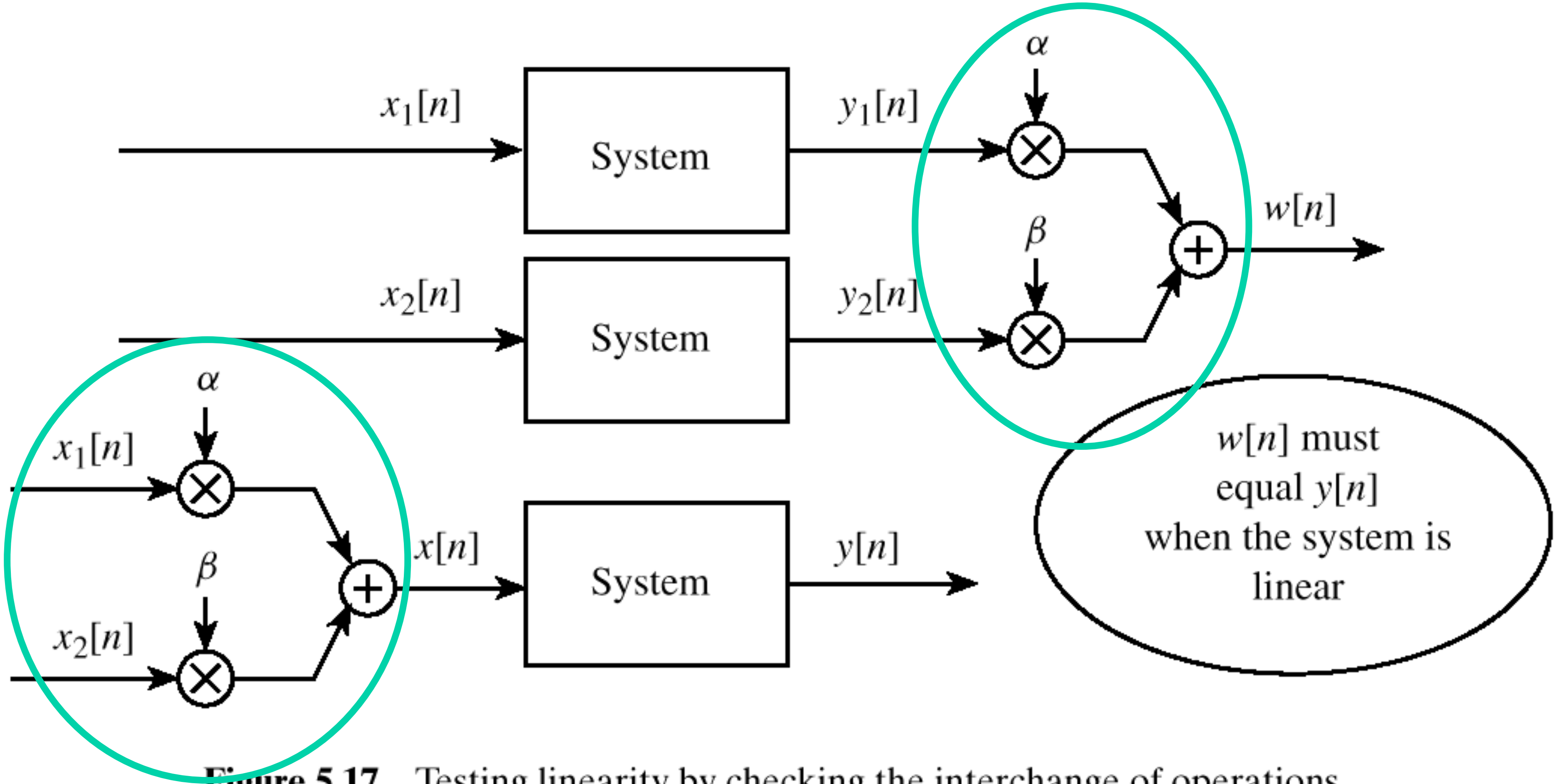


Figure 5.17 Testing linearity by checking the interchange of operations.

LTI SYSTEMS

- LTI: Linear & Time-Invariant
- COMPLETELY CHARACTERIZED by:
  - **IMPULSE RESPONSE** h[n]
  - **CONVOLUTION**: y[n] = x[n]*h[n]
    - The "rule"defining the system can ALWAYS be re-written as convolution
- FIR Example: h[n] is same as b_k

Pop Quiz

- FIR Filter is "FIRST DIFFERENCE"
  - $y[n] = x[n] - x[n-1]$
- Write output as a convolution
  - Need impulse response

$$h[n] = \delta[n] - \delta[n-1]$$

  - Then, another way to compute the output:

$$y[n] = \left(\delta[n] - \delta[n-1]\right) * x[n]$$

Cascade Systems

- Does the order of S_1 & S_2 matter?
  - NO, LTI SYSTEMS can be rearranged !!!
  - WHAT ARE THE FILTER COEFFS? $\{b_k\}$

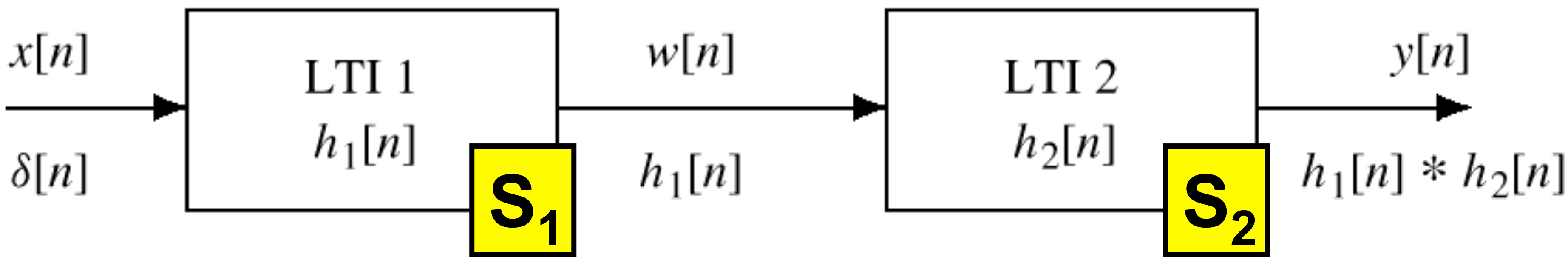


Figure 5.19 A Cascade of Two LTI Systems.

Cascade Equivalent

– Find "overall" h[n] for a cascade ?

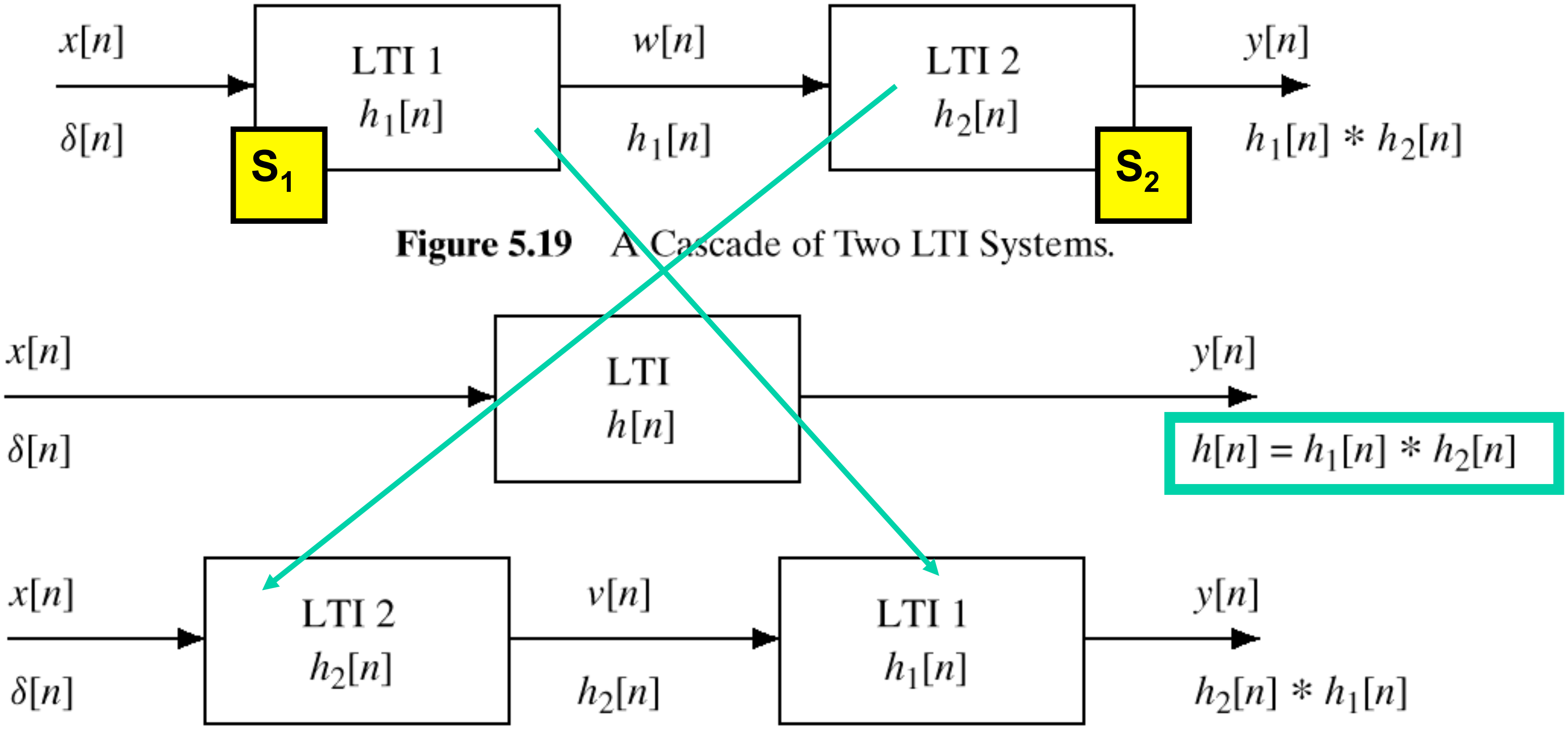


Figure 5.20 Switching the order of cascaded LTI systems.

- Next week <> Section 6-1
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