ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Circuits and Systems I

LECTURE #2
Phasor Addition

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License Info for SPFirst Slides

Outline - Today

- Today <> Section 2-6
  Lab 1!

- Next week <> Section 3-1
  Section 3-2
  Section 3-3
  Section 3-7
  Section 3-8

- Recommended self-study next week +
  - Appendix A: Complex Numbers ***read***
  - Appendix B: MATLAB ***read***

Lecture Objectives

- Phasors = Complex Amplitude
  - Complex Numbers **represent** Sinusoids

$$z(t) = Xe^{j\omega t} = (Ae^{j\varphi})e^{j\omega t}$$

- Develop the ABSTRACTION:
  - Adding Sinusoids = Complex Addition
  - **PHASOR ADDITION THEOREM**

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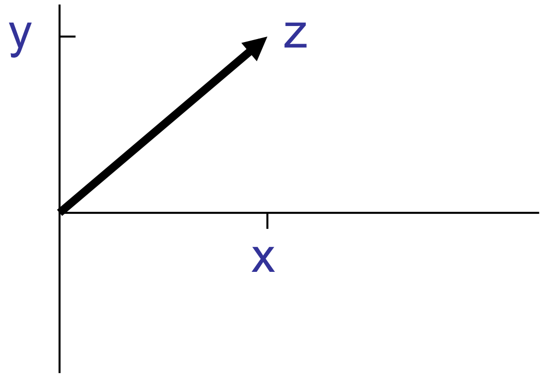
***CSI* Progress Level:**

Do You Remember The Complex Numbers?

Lab 2 warm-up

- To solve: $z^2 = -1$
  - $z = \boldsymbol{j}$
  - Math and Physics use $z = \boldsymbol{i}$
- Complex number: $z = x + \boldsymbol{j}\, y$

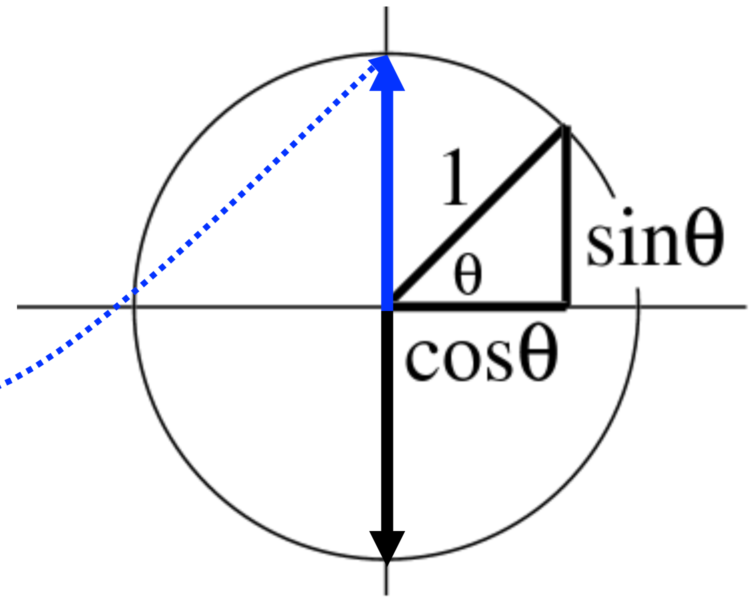


Cartesian coordinate system

Polar Form

- Vector Form
  - **Length** =1
  - **Angle** = θ

- Common Values
  - **j** has angle of 0.5π
  - -1 has angle of π
  - **−j** has angle of 1.5π
  - also, angle of **−j could** be $-0.5\pi = 1.5\pi - 2\pi$
  - because the PHASE is **AMBIGUOUS**

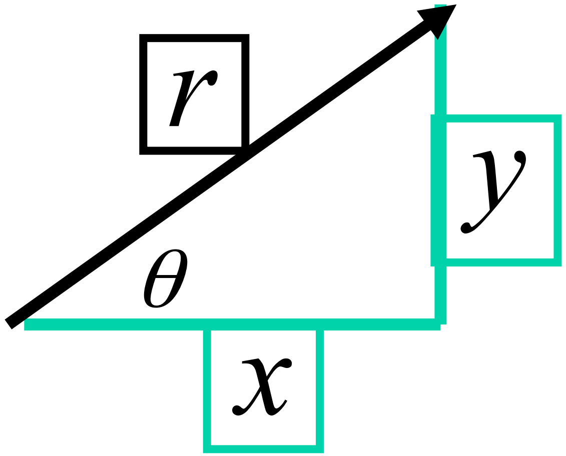


Polar <> Rectangular

- Relate (x,y) to (r,θ)

$$r^2 = x^2 + y^2$$

$$\theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right)$$



$$x = r\cos\theta$$

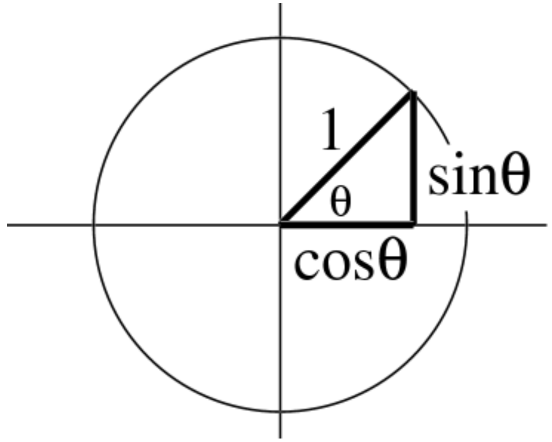
$$y = r\sin\theta$$

Most calculators do Polar-Rectangular

Need a notation for POLAR FORM

Euler's Formula

- **Complex Exponential**
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one



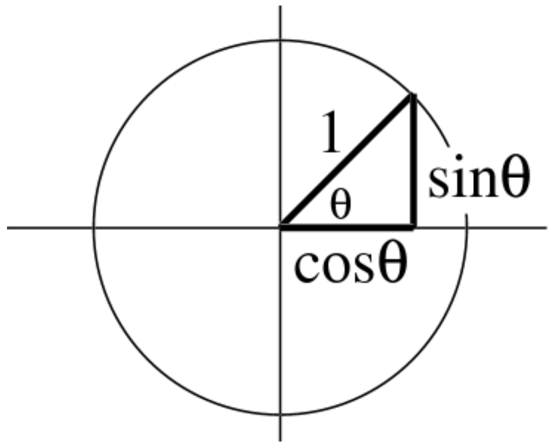
$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

Complex Exponential

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- Interpret this as a **Rotating Vector**
  - $\theta = \omega t$
  - Angle changes vs. time
  - ex: $\omega = 20\pi$ rad/s
  - Rotates 0.2π in 0.01 secs



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

cos = Real Part

Real Part of Euler's $\cos(\omega t) = \Re e\{e^{j\omega t}\}$

General Sinusoid $x(t) = A\cos(\omega t + \varphi)$

So, $$A\cos(\omega t + \varphi) = \Re e\{Ae^{j(\omega t + \varphi)}\}$$
$$= \Re e\{Ae^{j\varphi}e^{j\omega t}\}$$

Real Part Example

$$A\cos(\omega t+\varphi)=\Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Evaluate: $x(t)=\Re e\left\{-3je^{j\omega t}\right\}$

Answer:

$$x(t)=\Re e\left\{(-3j)e^{j\omega t}\right\}$$

$$=\Re e\left\{3e^{-j0.5\pi}e^{j\omega t}\right\}=3\cos(\omega t-0.5\pi)$$

Complex Amplitude

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

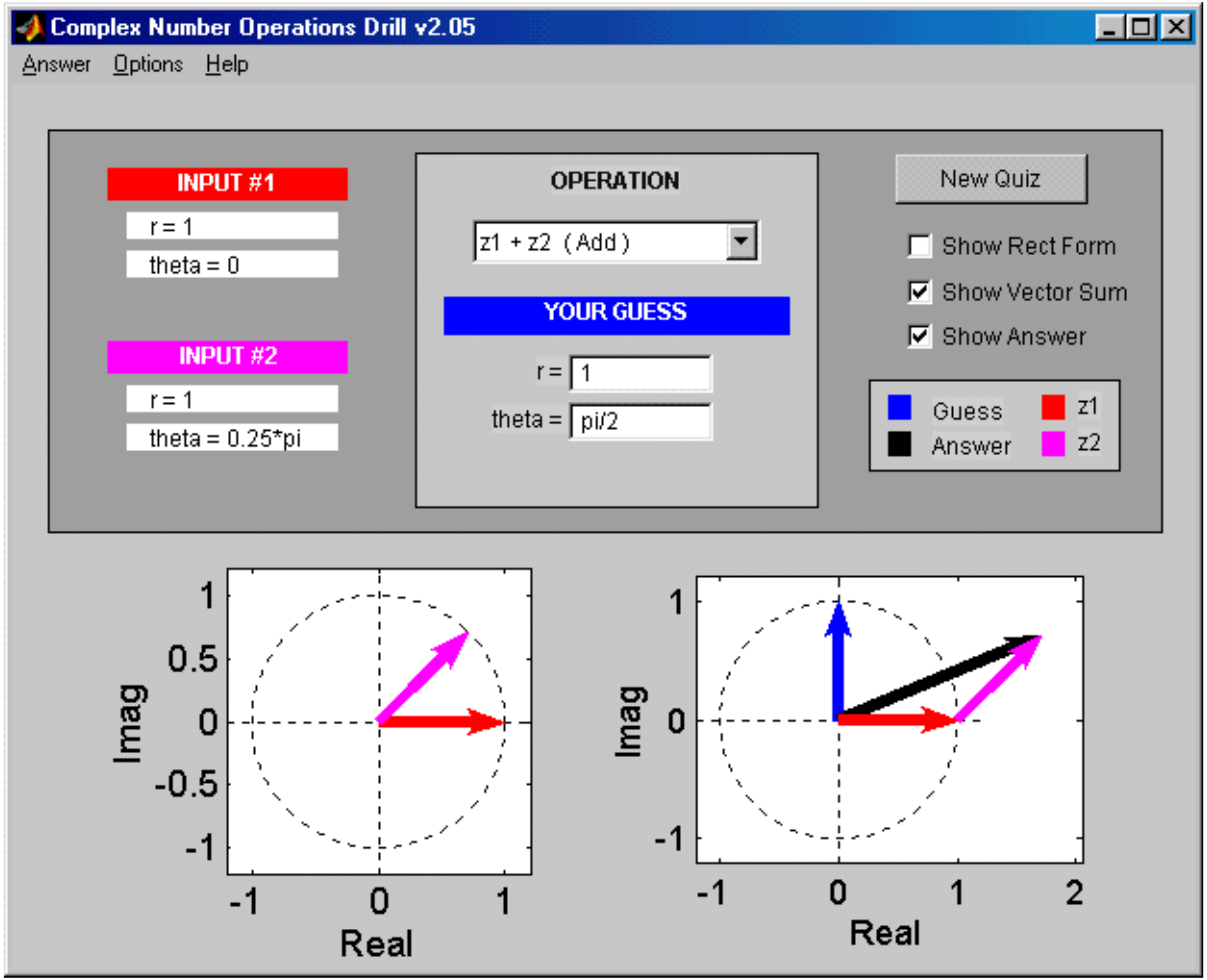
Complex AMPLITUDE = X

$$z(t) = Xe^{j\omega t} \qquad X = Ae^{j\varphi}$$

Then, any Sinusoid = REAL PART of $Xe^{j\omega t}$

$$x(t) = \Re e\left\{Xe^{j\omega t}\right\} = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Z DRILL (Complex Arith)

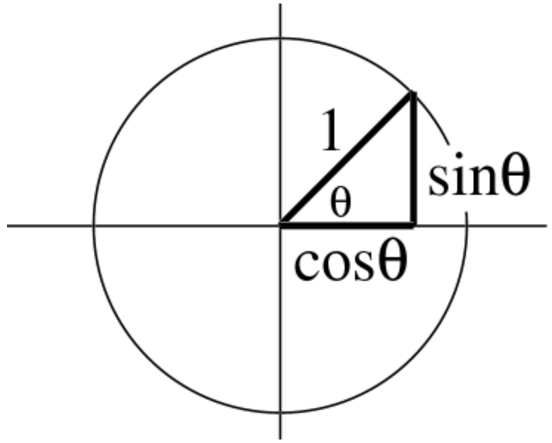


How to AVOID Trigonometry

- Algebra, even complex, is EASIER !!!
- Can you recall $\cos(\theta_1+\theta_2)$?
- Use: real part of $e^{j(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2)$

Recall Euler's FORMULA

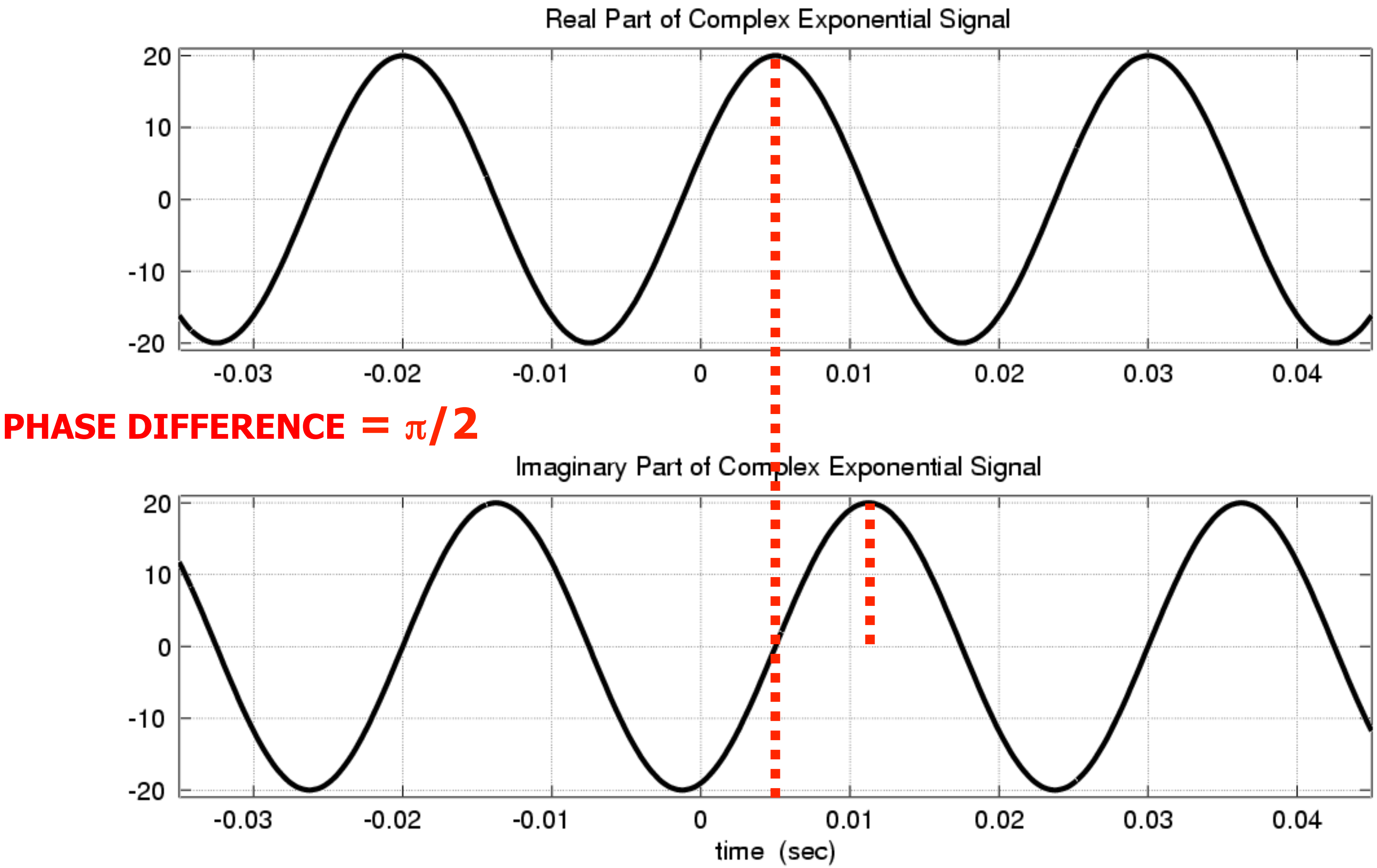
- Complex Exponential
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\boxed{e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)}$$

Real & Imaginary Part Plots



Complex Exponential

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- Interpret this as a **Rotating Vector**
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  - ex: $\omega = 20\pi$ rad/s
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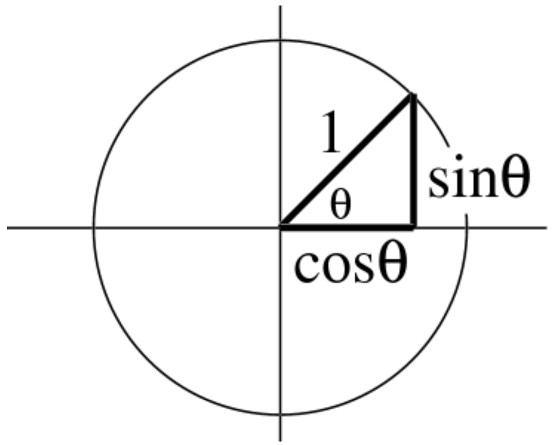
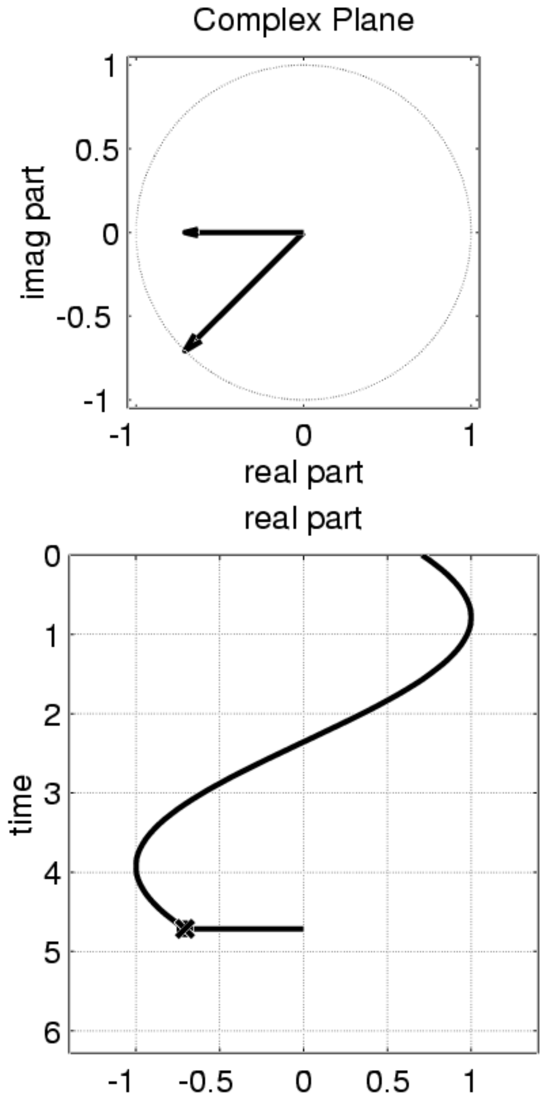


$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Rotating Phasor

See Demo on CD-ROM
Chapter 2



Cos = REAL PART

Real Part of Euler's

$$\cos(\omega t) = \Re e\left\{e^{j\omega t}\right\}$$

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi)$$

So,

$$A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j(\omega t + \varphi)}\right\}$$

$$= \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Complex Amplitude

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Sinusoid = REAL PART of $(Ae^{j\phi})e^{j\omega t}$

$$x(t) = \Re e\left\{Xe^{j\omega t}\right\} = \Re e\{z(t)\}$$

Complex AMPLITUDE = X

$$z(t) = Xe^{j\omega t} \qquad X = Ae^{j\varphi}$$

POP QUIZ: Complex Amp

- Find the COMPLEX AMPLITUDE for:

$$x(t) = \sqrt{3}\cos(77\pi t + 0.5\pi)$$

- Use EULER's FORMULA:

$$x(t) = \Re e\left\{\sqrt{3}e^{j(77\pi t + 0.5\pi)}\right\}$$

$$= \Re e\left\{\sqrt{3}e^{j0.5\pi}e^{j77\pi t}\right\}$$

$$X = \sqrt{3}e^{j0.5\pi}$$

Want to Add Sinusoids

- ALL SINUSOIDS have SAME FREQUENCY
- HOW to GET {Amp,Phase} of RESULT ?

$$x_1(t) = 1.7\cos(2\pi(10)t + 70\pi/180)$$

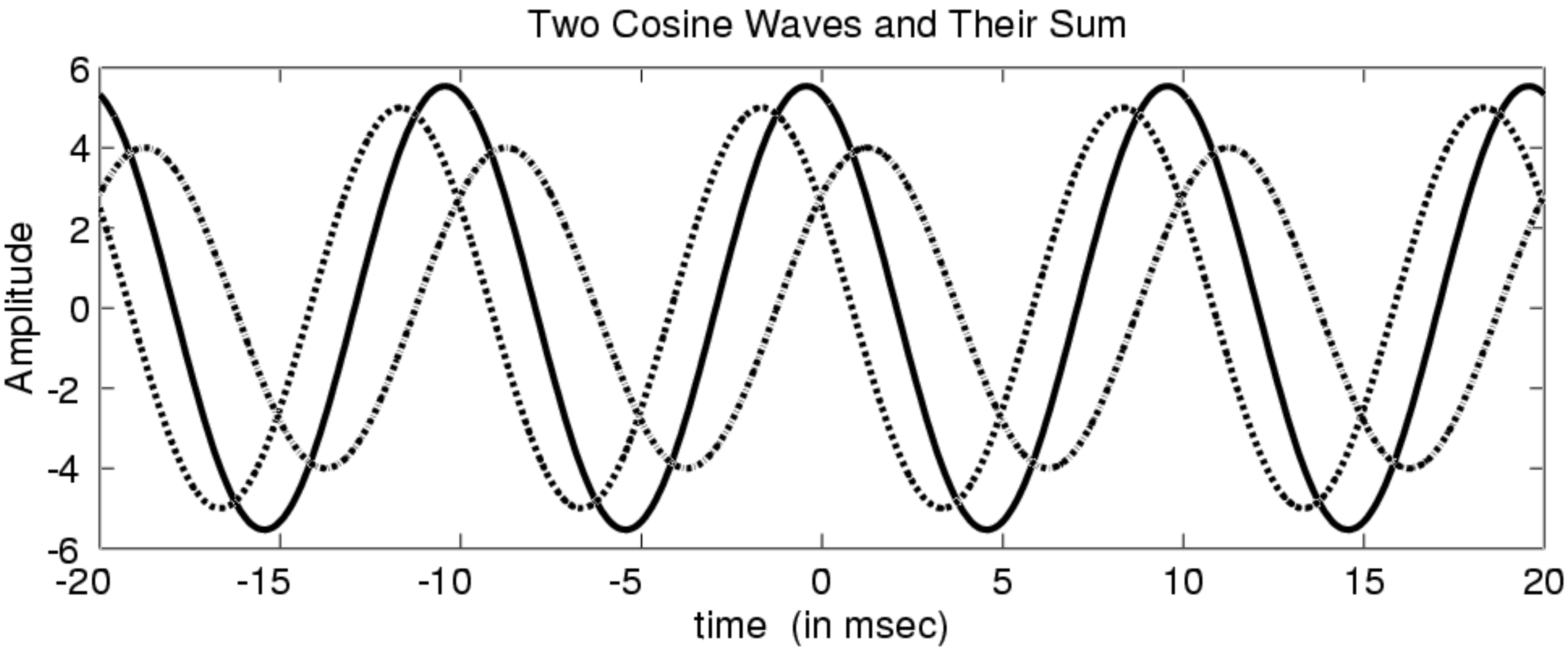
$$x_2(t) = 1.9\cos(2\pi(10)t + 200\pi/180)$$

$$x_3(t) = x_1(t) + x_2(t)$$

$$= 1.532\cos(2\pi(10)t + 141.79\pi/180)$$

Add Sinusoids

- Sum Sinusoid has **SAME** Frequency



Phasor Addition Rule

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k)$$

$$= A \cos(\omega_0 t + \phi)$$

Get the new complex amplitude by complex addition

$$\sum_{k=1}^{N} A_k e^{j\phi_k} = A e^{j\phi}$$

Phasor Addition Proof

$$\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = \sum_{k=1}^{N} \Re e\left\{A_k e^{j(\omega_0 t + \phi_k)}\right\}$$

$$= \Re e\left\{\sum_{k=1}^{N} A_k e^{j\phi_k} e^{j\omega_0 t}\right\}$$

$$= \Re e\left\{\left(\sum_{k=1}^{N} A_k e^{j\phi_k}\right) e^{j\omega_0 t}\right\}$$

$$= \Re e\left\{\left(A e^{j\phi}\right) e^{j\omega_0 t}\right\} = A\cos(\omega_0 t + \phi)$$

POP QUIZ: Add Sinusoids

- ADD THESE 2 SINUSOIDS:

$$x_1(t) = \cos(77\pi t)$$

$$x_2(t) = \sqrt{3}\cos(77\pi t + 0.5\pi)$$

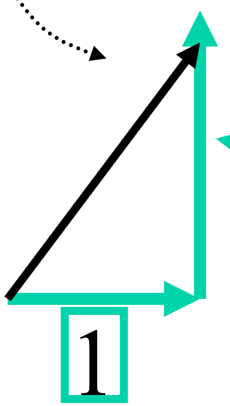
- COMPLEX ADDITION:

$$1e^{j0} + \sqrt{3}e^{j0.5\pi}$$

POP QUIZ (answer)

- COMPLEX ADDITION:

$$1 + j\sqrt{3} = 2e^{j\pi/3}$$

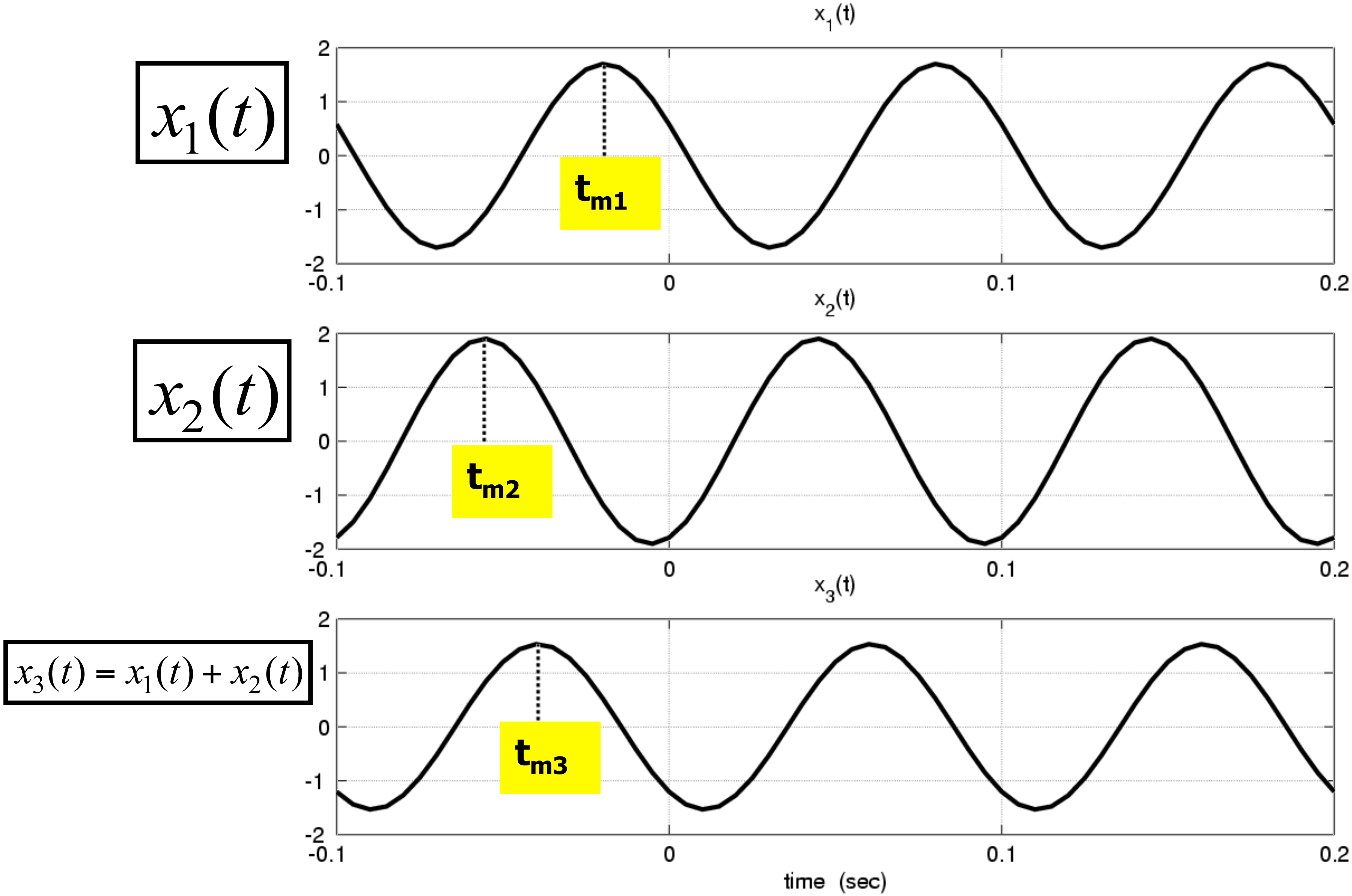


$$j \cdot \sqrt{3} = \sqrt{3}e^{j0.5\pi}$$

- CONVERT back to cosine form:

$$x_3(t) = 2\cos(77\pi t + \frac{\pi}{3})$$

Add Sinusoids Example

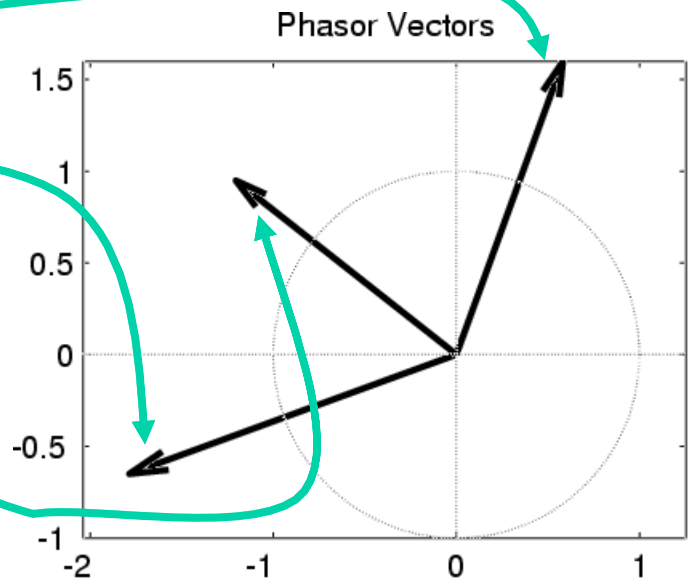


Convert Time-Shift to Phase

- Measure peak times:
  - t_{m1}=-0.0194, t_{m2}=-0.0556, t_{m3}=-0.0394
- Convert to phase (T=0.1)
  - $\phi_1 = -\omega t_{m1} = -2\pi(t_{m1}/T) = 70\pi/180$,
  - $\phi_2 = 200\pi/180$
- Amplitudes
  - A_1=1.7, A_2=1.9, A_3=1.532

Phasor Add: Numerical

- Convert Polar to Cartesian
  - $X_1 = 0.5814 + j1.597$
  - $X_2 = -1.785 - j0.6498$
  - sum =
  - $X_3 = -1.204 + j0.9476$
- Convert back to Polar
  - $X_3 = 1.532$ at angle $141.79\pi/180$
  - This is the sum



Add Sinusoids

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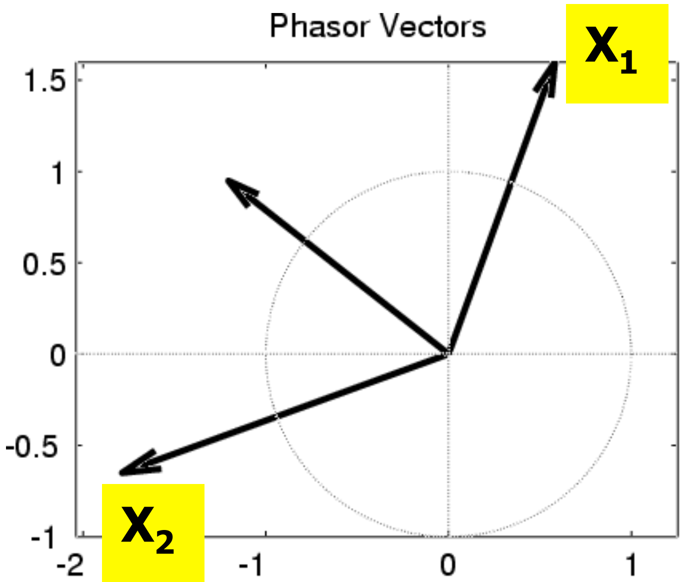
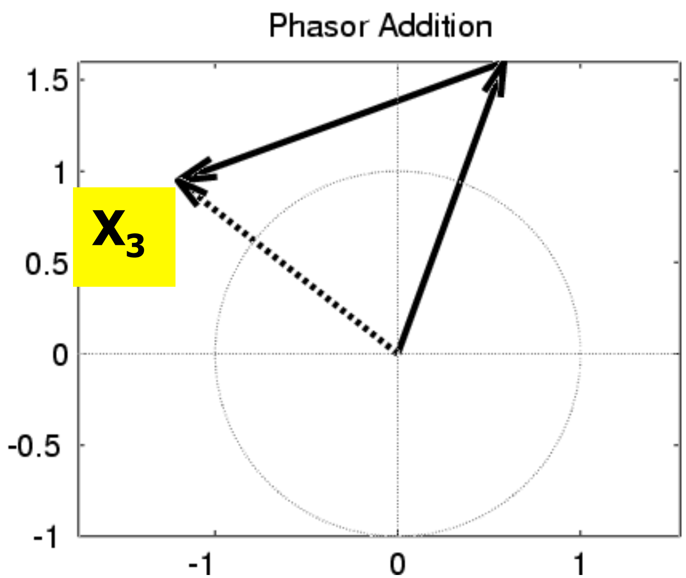
VECTOR (PHASOR) ADD





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LAB TIME NOW!