



ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Circuits and Systems I

LECTURE #2

Phasor Addition

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Outline - Today

- Today <> Section 2-6
Lab 1!
- Next week <> Section 3-1
Section 3-2
Section 3-3
Section 3-7
Section 3-8
- Recommended self-study next week +
 - Appendix A: Complex Numbers **read**
 - Appendix B: MATLAB **read**

Lecture Objectives

- Phasors = Complex Amplitude
 - Complex Numbers **represent** Sinusoids

$$z(t) = Xe^{j\omega t} = (Ae^{j\varphi})e^{j\omega t}$$

- Develop the ABSTRACTION:
 - Adding Sinusoids = Complex Addition
 - **PHASOR ADDITION THEOREM**

Lecture Objectives

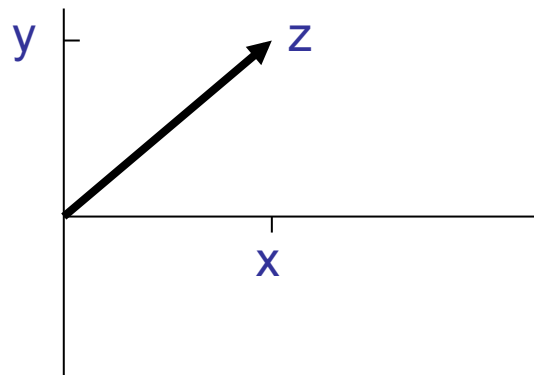
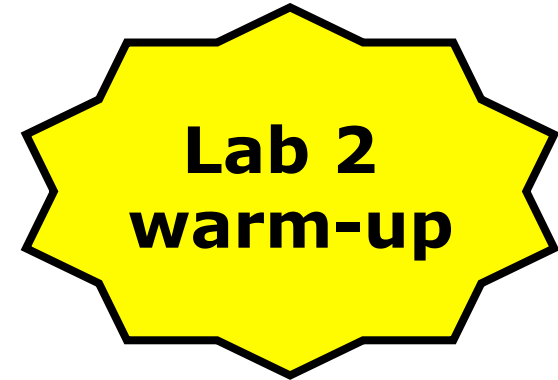
- Phasors = Complex Amplitude
 - Complex Numbers **represent** Sinusoids
- Develop the ABSTRACTION:
 - Adding Sinusoids = Complex Addition
 - **PHASOR ADDITION THEOREM**

CSI
Progress
Level:



Do You Remember The Complex Numbers?

- To solve: $z^2 = -1$
 - $z = \mathbf{j}$
 - Math and Physics use $z = \mathbf{i}$
- Complex number: $z = x + \mathbf{j} y$



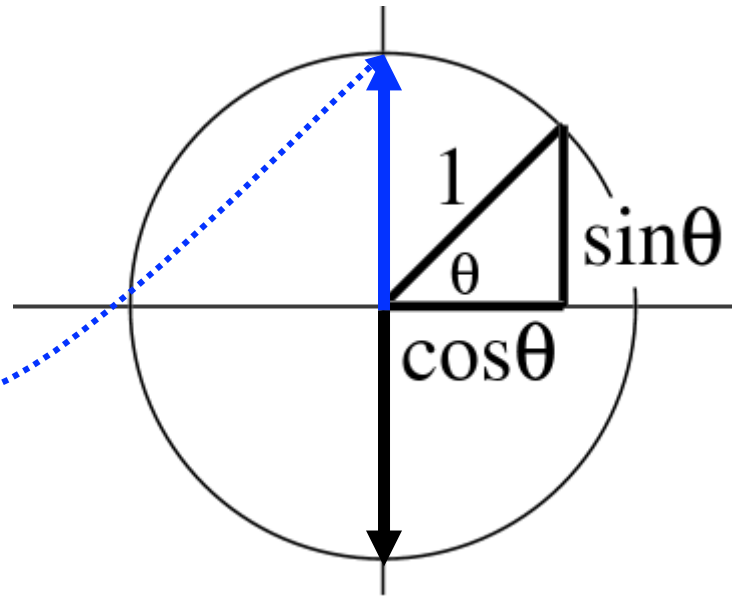
Cartesian
coordinate
system

Polar Form

- Vector Form
 - **Length** = 1
 - **Angle** = θ

- Common Values

- **j** has angle of 0.5π
- -1 has angle of π
- $-\mathbf{j}$ has angle of 1.5π
- also, angle of $-\mathbf{j}$ **could** be $-0.5\pi = 1.5\pi - 2\pi$
- because the PHASE is **AMBIGUOUS**



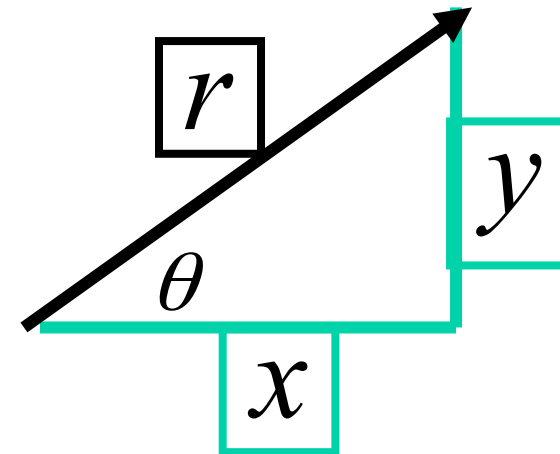
Polar <>

Rectangular

- Relate (x,y) to (r,θ)

$$r^2 = x^2 + y^2$$
$$\theta = \text{Tan}^{-1}\left(\frac{y}{x}\right)$$

Most calculators do
Polar-Rectangular



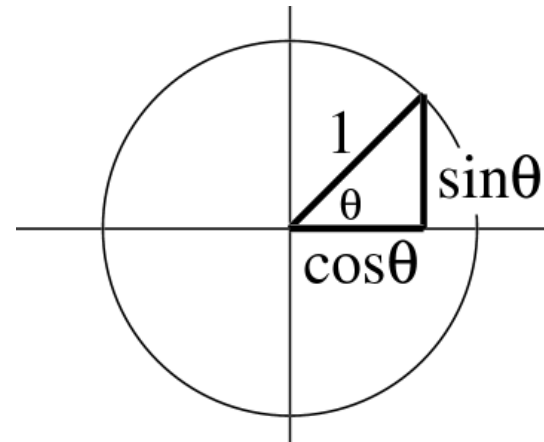
$$x = r \cos \theta$$
$$y = r \sin \theta$$

Need a notation for POLAR FORM

Euler's Formula

- **Complex Exponential**

- Real part is cosine
- Imaginary part is sine
- Magnitude is one



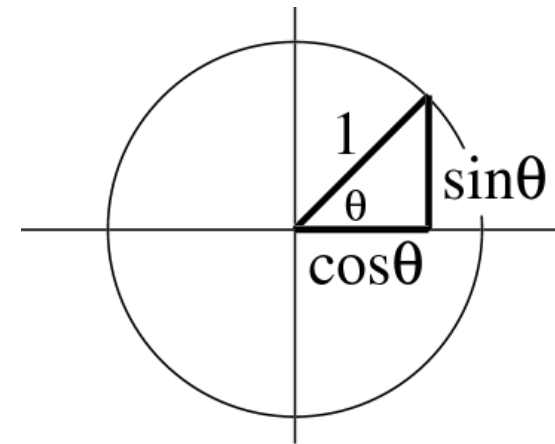
$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$re^{j\theta} = r \cos(\theta) + jr \sin(\theta)$$

Complex Exponential

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

- Interpret this as a **Rotating Vector**
 - $\theta = \omega t$
 - Angle changes vs. time
 - ex: $\omega = 20\pi$ rad/s
 - Rotates 0.2π in 0.01 secs



$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

cos = Real Part

Real Part of Euler's $\cos(\omega t) = \Re\{e^{j\omega t}\}$

General Sinusoid $x(t) = A \cos(\omega t + \varphi)$

So, $A \cos(\omega t + \varphi) = \Re\{Ae^{j(\omega t + \varphi)}\}$
 $= \Re\{Ae^{j\varphi} e^{j\omega t}\}$

Real Part Example

$$A \cos(\omega t + \varphi) = \Re \left\{ A e^{j\varphi} e^{j\omega t} \right\}$$

Evaluate: $x(t) = \Re \left\{ -3j e^{j\omega t} \right\}$

Answer:

$$\begin{aligned} x(t) &= \Re \left\{ (-3j) e^{j\omega t} \right\} \\ &= \Re \left\{ 3 e^{-j0.5\pi} e^{j\omega t} \right\} = 3 \cos(\omega t - 0.5\pi) \end{aligned}$$

Complex Amplitude

General Sinusoid

$$x(t) = A \cos(\omega t + \varphi) = \Re\{Ae^{j\varphi} e^{j\omega t}\}$$

Complex AMPLITUDE = X

$z(t) = Xe^{j\omega t}$	$X = Ae^{j\varphi}$
-------------------------	---------------------

Then, any Sinusoid = REAL PART of $Xe^{j\omega t}$

$$x(t) = \Re\{Xe^{j\omega t}\} = \Re\{Ae^{j\varphi} e^{j\omega t}\}$$

Z DRILL (Complex Arith)

Complex Number Operations Drill v2.05 _ □ ×

[Answer](#) [Options](#) [Help](#)

INPUT #1

r = 1

theta = 0

INPUT #2

r = 1

theta = 0.25*pi

OPERATION

z1 + z2 (Add) ▾

YOUR GUESS

r = 1

theta = pi/2

New Quiz

Show Rect Form

Show Vector Sum

Show Answer

	Guess		z1
	Answer		z2

Imag

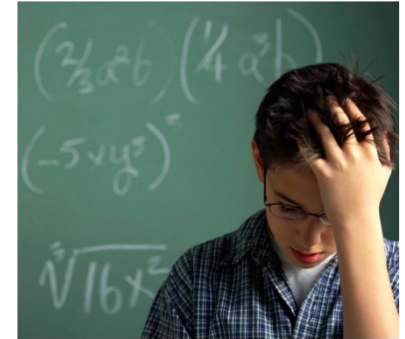
Real

Imag

Real

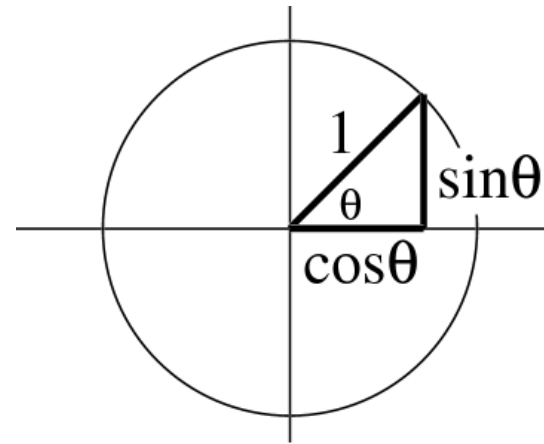
How to AVOID Trigonometry

- Algebra, even complex, is **EASIER** !!!
- Can you recall $\cos(\theta_1 + \theta_2)$?
- Use: real part of $e^{j(\theta_1 + \theta_2)} = \cos(\theta_1 + \theta_2)$



Recall Euler's FORMULA

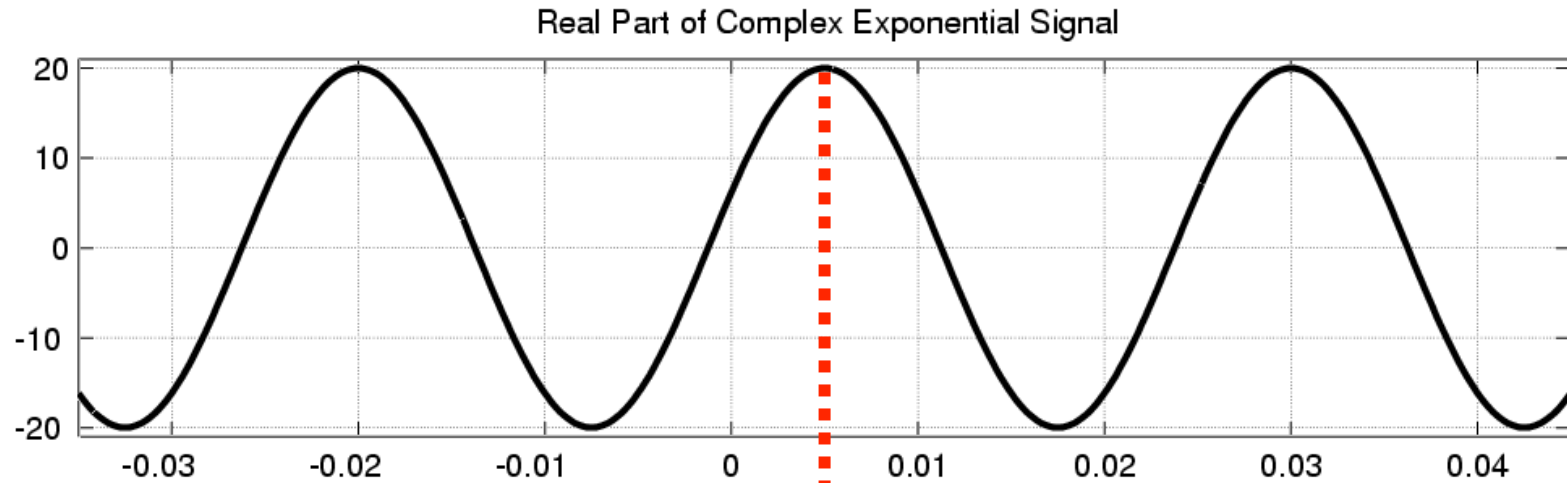
- Complex Exponential
 - Real part is cosine
 - Imaginary part is sine
 - Magnitude is one



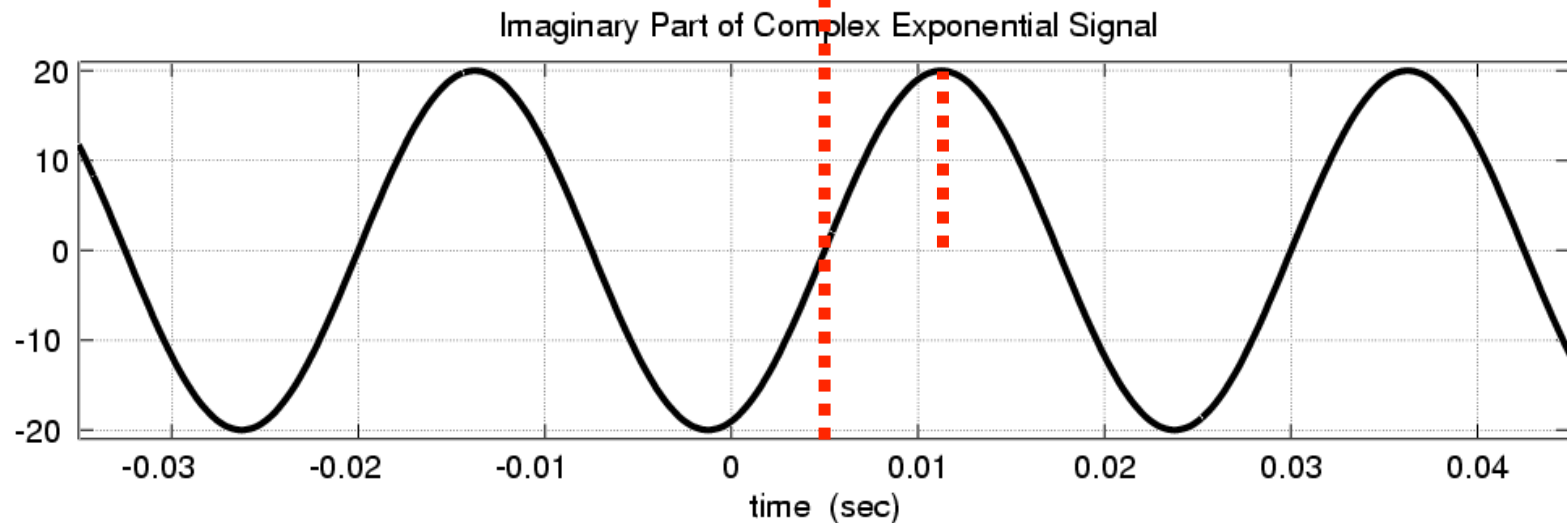
$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

Real & Imaginary Part Plots



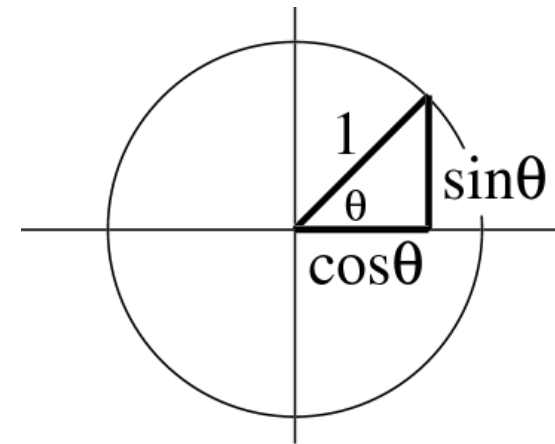
PHASE DIFFERENCE = $\pi/2$



Complex Exponential

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

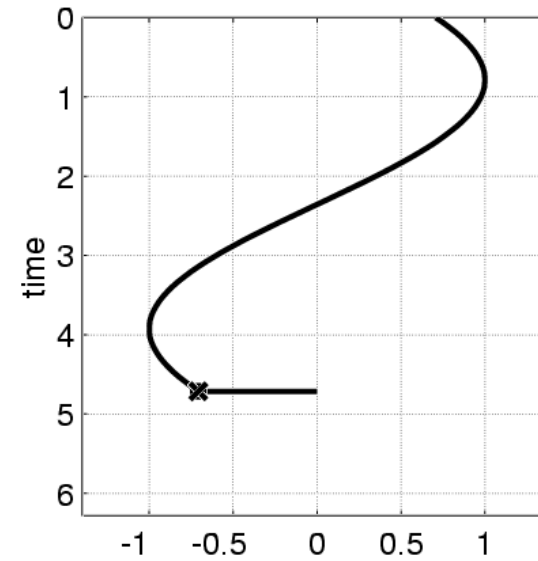
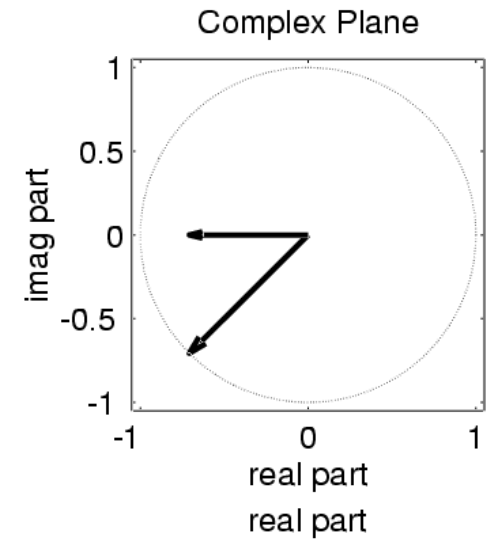
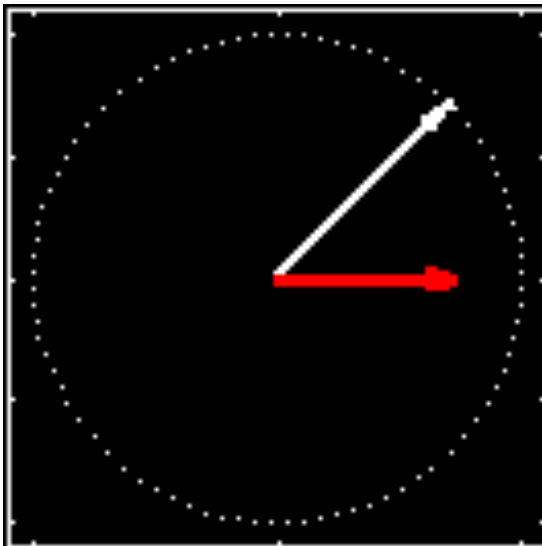
- Interpret this as a **Rotating Vector**
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$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

Rotating Phasor

See Demo on CD-ROM
Chapter 2



Cos = REAL PART

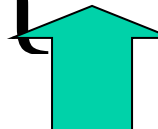
Real Part of Euler's

$$\cos(\omega t) = \Re\{e^{j\omega t}\}$$

General Sinusoid

$$x(t) = A \cos(\omega t + \varphi)$$

So,

$$\begin{aligned} A \cos(\omega t + \varphi) &= \Re\{Ae^{j(\omega t + \varphi)}\} \\ &= \Re\{Ae^{j\varphi} e^{j\omega t}\} \end{aligned}$$


Complex Amplitude

General Sinusoid

$$x(t) = A \cos(\omega t + \varphi) = \Re \{ A e^{j\varphi} e^{j\omega t} \}$$

Sinusoid = REAL PART of $(Ae^{j\varphi})e^{j\omega t}$

$$x(t) = \Re \{ X e^{j\omega t} \} = \Re \{ z(t) \}$$

Complex AMPLITUDE = X

$$z(t) = X e^{j\omega t} \quad X = A e^{j\varphi}$$

POP QUIZ: Complex Amp

- Find the COMPLEX AMPLITUDE for:

$$x(t) = \sqrt{3} \cos(77\pi t + 0.5\pi)$$

- Use EULER'S FORMULA:

$$\begin{aligned} x(t) &= \Re\left\{ \sqrt{3} e^{j(77\pi t + 0.5\pi)} \right\} \\ &= \Re\left\{ \sqrt{3} e^{j0.5\pi} e^{j77\pi t} \right\} \end{aligned}$$

$$X = \sqrt{3} e^{j0.5\pi}$$

Want to Add Sinusoids

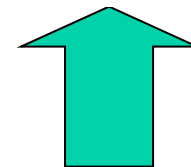
- ALL SINUSOIDS have **SAME** FREQUENCY
- HOW to GET **{Amp,Phase}** of RESULT ?

$$x_1(t) = 1.7 \cos(2\pi(10)t + 70\pi/180)$$

$$x_2(t) = 1.9 \cos(2\pi(10)t + 200\pi/180)$$

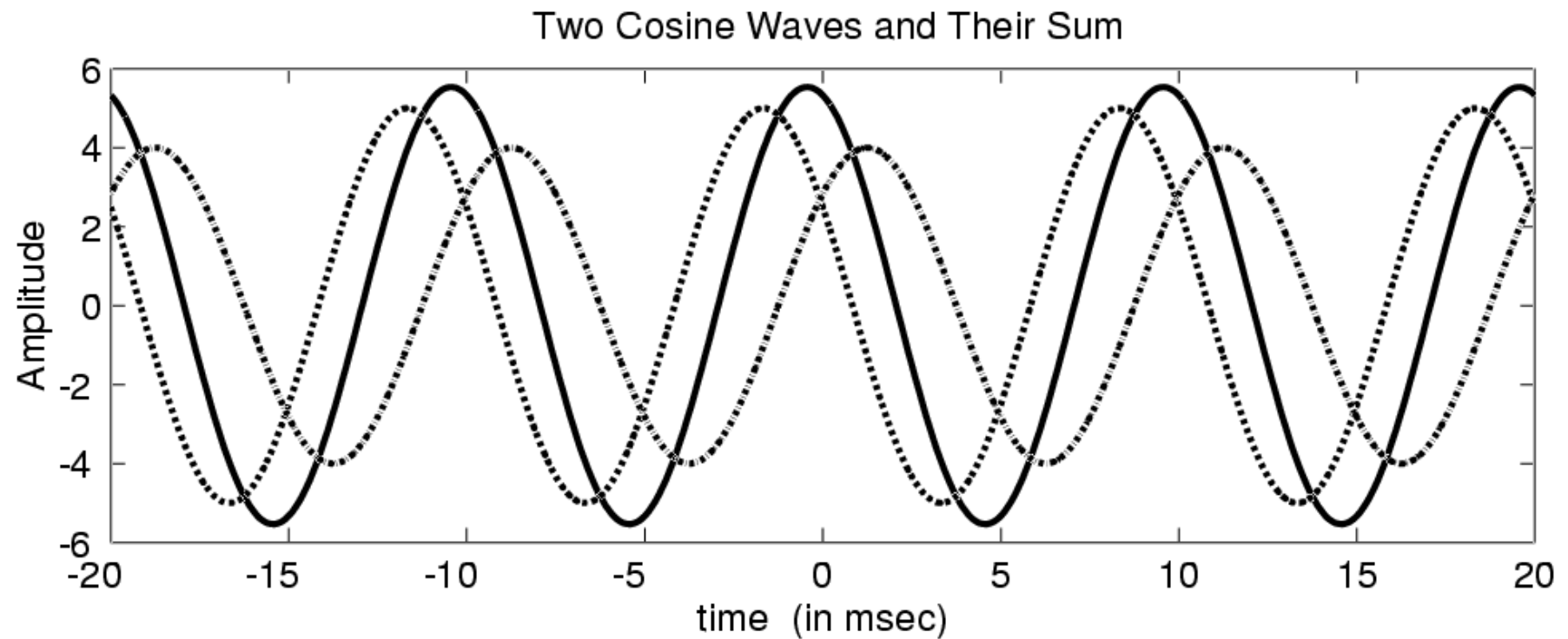
$$x_3(t) = x_1(t) + x_2(t)$$

$$= 1.532 \cos(2\pi(10)t + 141.79\pi/180)$$



Add Sinusoids

- Sum Sinusoid has SAME Frequency



Phasor Addition Rule

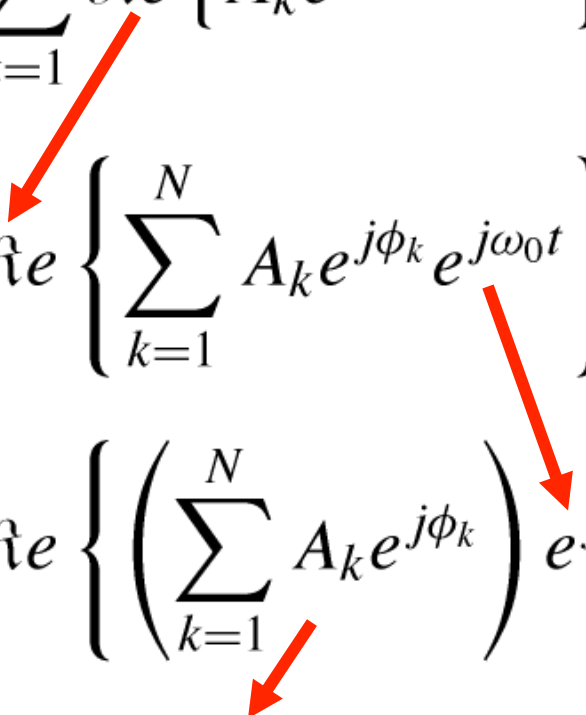
$$x(t) = \sum_{k=1}^N A_k \cos(\omega_0 t + \phi_k)$$

$$= A \cos(\omega_0 t + \phi)$$

Get the new complex amplitude by complex addition

$$\sum_{k=1}^N A_k e^{j\phi_k} = A e^{j\phi}$$

Phasor Addition Proof

$$\begin{aligned}\sum_{k=1}^N A_k \cos(\omega_0 t + \phi_k) &= \sum_{k=1}^N \Re e \left\{ A_k e^{j(\omega_0 t + \phi_k)} \right\} \\ &= \Re e \left\{ \sum_{k=1}^N A_k e^{j\phi_k} e^{j\omega_0 t} \right\} \\ &= \Re e \left\{ \left(\sum_{k=1}^N A_k e^{j\phi_k} \right) e^{j\omega_0 t} \right\} \\ &= \Re e \left\{ (A e^{j\phi}) e^{j\omega_0 t} \right\} = A \cos(\omega_0 t + \phi)\end{aligned}$$


POP QUIZ: Add Sinusoids

- ADD THESE 2 SINUSOIDS:

$$x_1(t) = \cos(77\pi t)$$

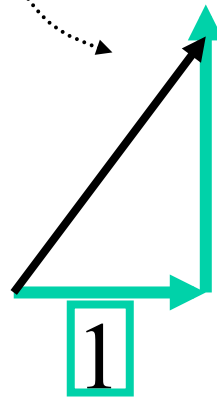
$$x_2(t) = \sqrt{3} \cos(77\pi t + 0.5\pi)$$

- COMPLEX ADDITION:

$$1e^{j0} + \sqrt{3}e^{j0.5\pi}$$

POP QUIZ (answer)

- COMPLEX ADDITION:



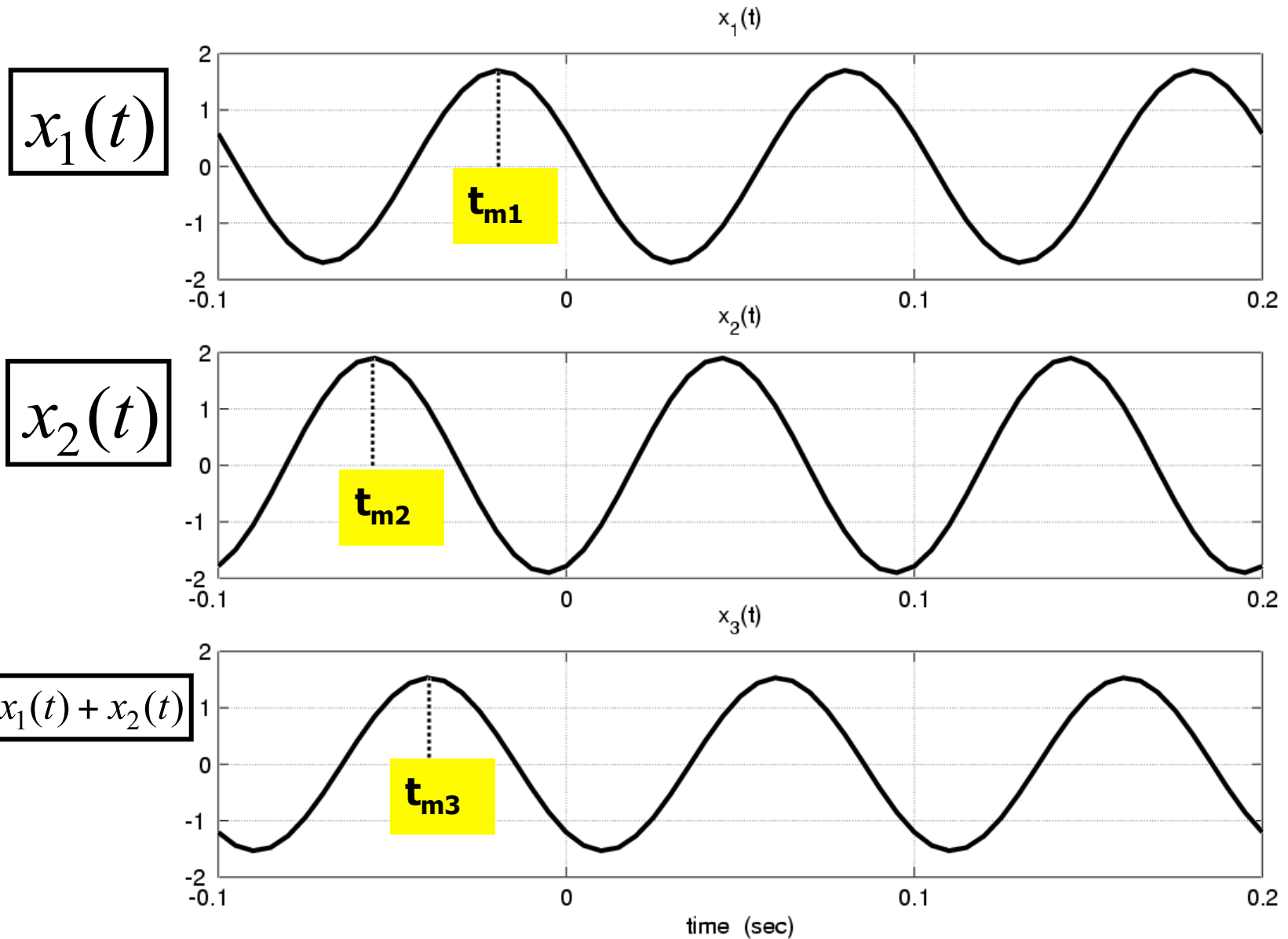
$$j\sqrt{3} = \sqrt{3}e^{j0.5\pi}$$

$$1 + j\sqrt{3} = 2e^{j\pi/3}$$

- CONVERT back to cosine form:

$$x_3(t) = 2 \cos\left(77\pi t + \frac{\pi}{3}\right)$$

Add Sinusoids Example



Convert Time-Shift to Phase

- Measure **peak times**:
 - $t_{m1} = -0.0194$, $t_{m2} = -0.0556$, $t_{m3} = -0.0394$
- Convert to **phase** ($T=0.1$)
 - $\phi_1 = -\omega t_{m1} = -2\pi(t_{m1}/T) = 70\pi/180$,
 - $\phi_2 = 200\pi/180$
- Amplitudes
 - $A_1 = 1.7$, $A_2 = 1.9$, $A_3 = 1.532$

Phasor Add: Numerical

- Convert Polar to Cartesian

- $X_1 = 0.5814 + j1.597$

- $X_2 = -1.785 - j0.6498$

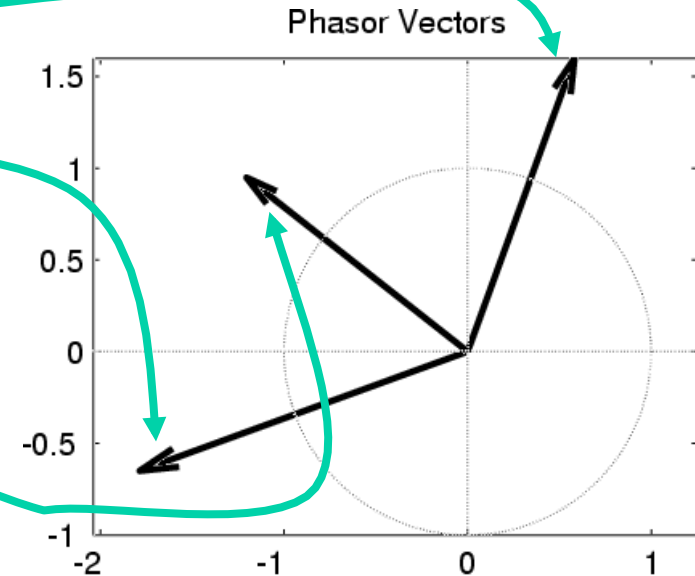
- sum =

- $X_3 = -1.204 + j0.9476$

- Convert back to Polar

- $X_3 = 1.532$ at angle $141.79\pi/180$

- This is the sum



Add Sinusoids

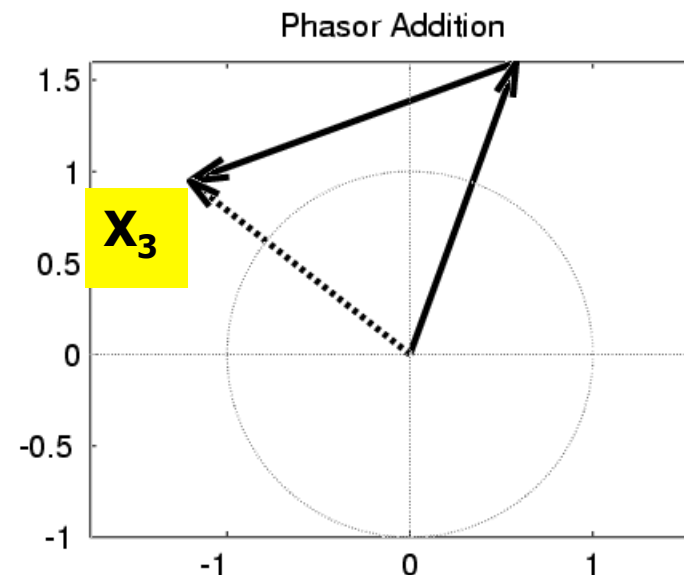
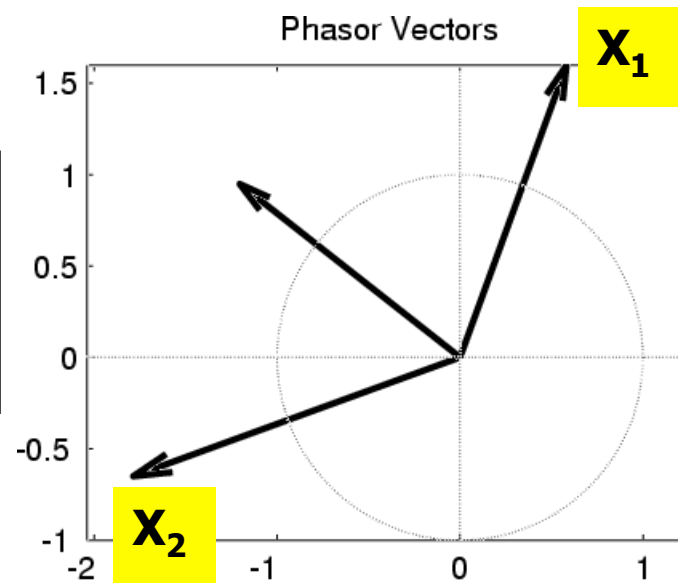
$$x_1(t) = 1.7 \cos(2\pi(10)t + 70\pi/180)$$

$$x_2(t) = 1.9 \cos(2\pi(10)t + 200\pi/180)$$

$$x_3(t) = x_1(t) + x_2(t)$$

$$= 1.532 \cos(2\pi(10)t + 141.79\pi/180)$$

VECTOR
(PHASOR)
ADD





That's all Folks!

- Next week

<>

Section 3-1
Section 3-2
Section 3-3
Section 3-7
Section 3-8

LAB TIME NOW!