ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Circuits and Systems I

LECTURE #3

The Spectrum, Periodic Signals, and the Time-Varying Spectrum

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License Info for SPFirst Slides

Outline - Today

- Today < > Section 3-1 – 3-3
  Section 3-7
  Section 3-8

Next week < > Section 3-4
Section 3-5
Section 3-6
Lab 2

READ (Sections 3-4, 3-5, 3-6 and Lab 2)

CSI
Progress
Level:

Lecture Objectives

- Sinusoids with DIFFERENT frequencies
  - SYNTHESIZE by Adding Sinusoids

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

- SPECTRUM Representation
  - Graphical Form shows **DIFFERENT** Freqs

CSI
Progress
Level:

FREQUENCY DIAGRAM

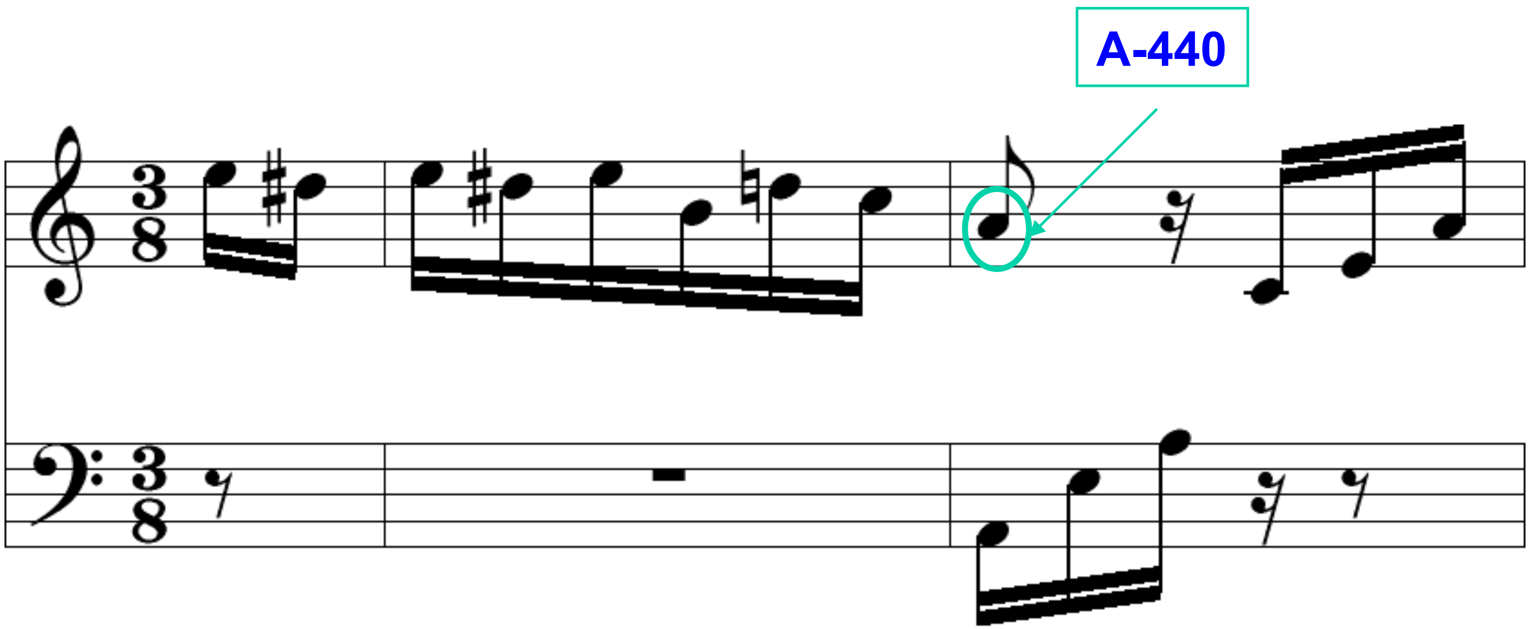


Figure 3.18 Sheet-music notation is a time–frequency diagram.

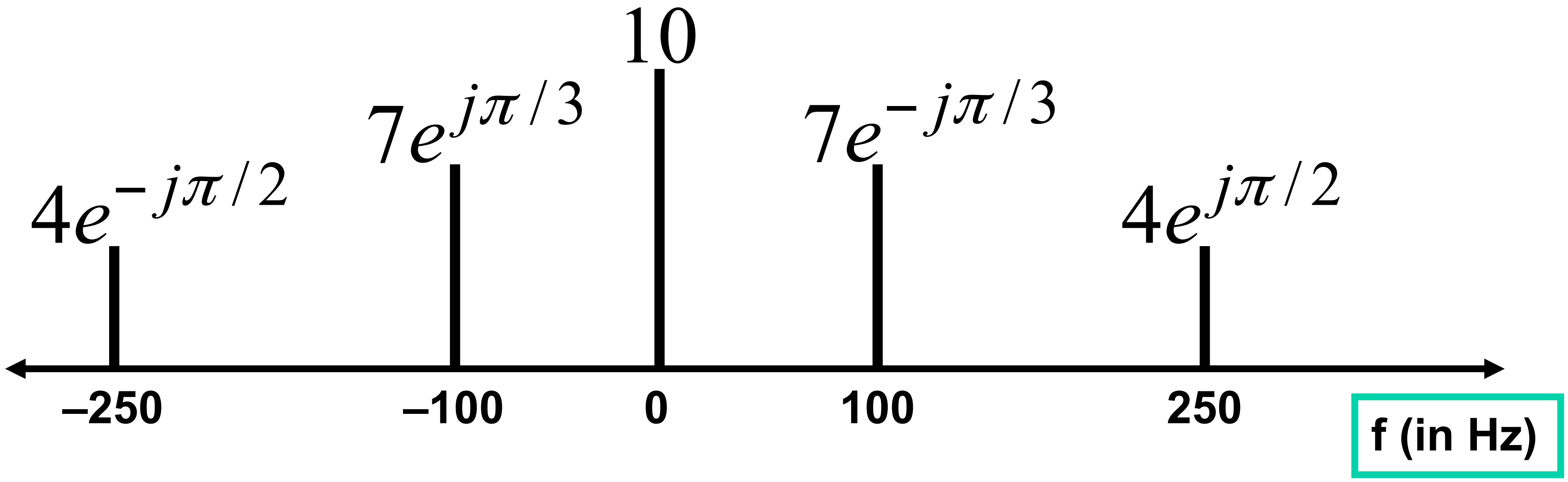
Frequency is the vertical axis

Time is the horizontal axis

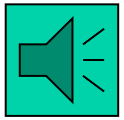
Another FREQ. Diagram

- Plot Complex Amplitude vs. Freq

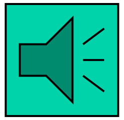
$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$



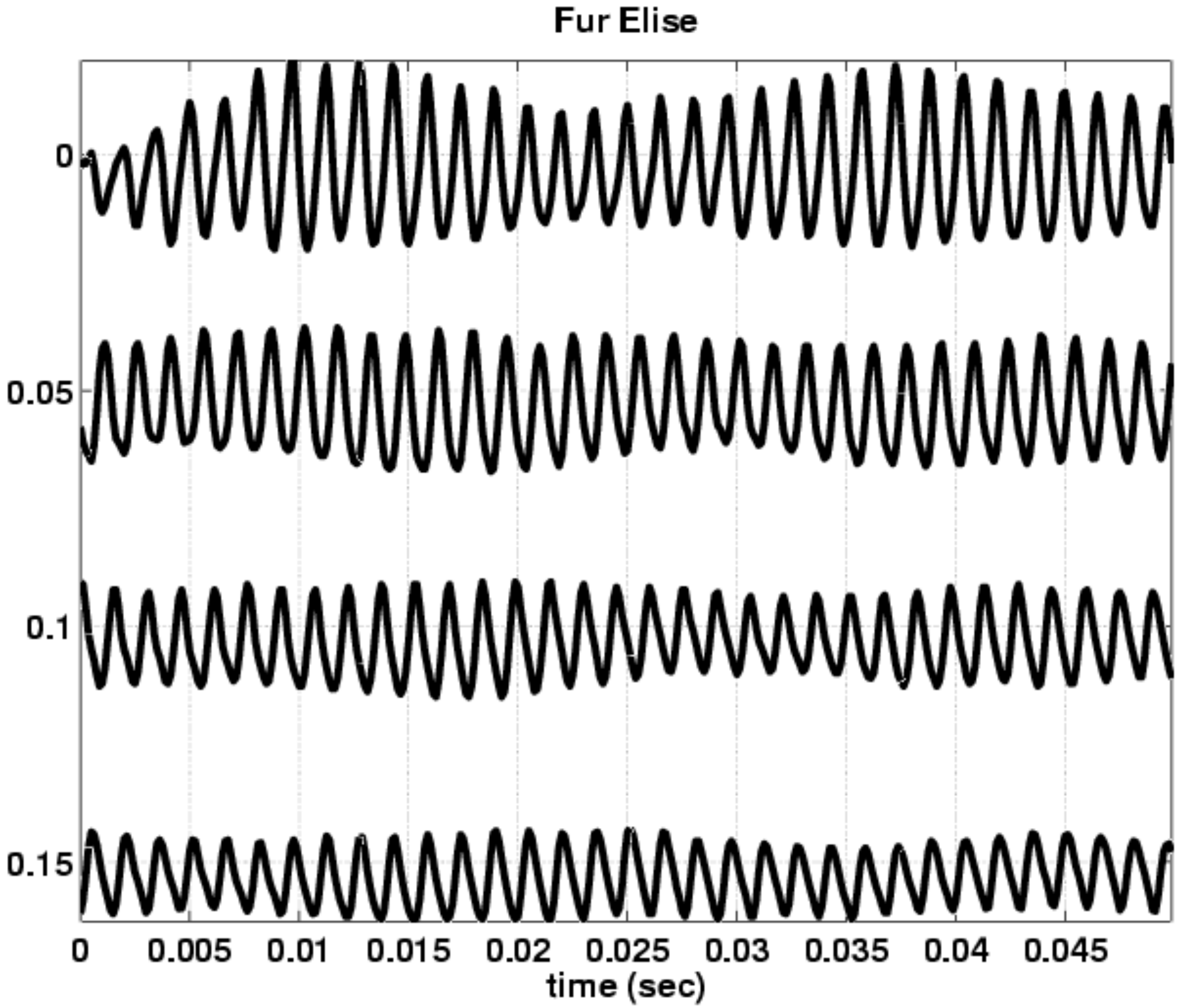
Motivation



- Synthesize Complicated Signals
  - Musical Notes
    - Piano uses 3 strings for many notes
    - Chords: play several notes simultaneously
  - Human Speech
    - Vowels have dominant frequencies
    - Application: computer generated speech
  - Can all signals be generated this way?
    - Sum of sinusoids?

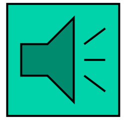


Fur Elise WAVEFORM

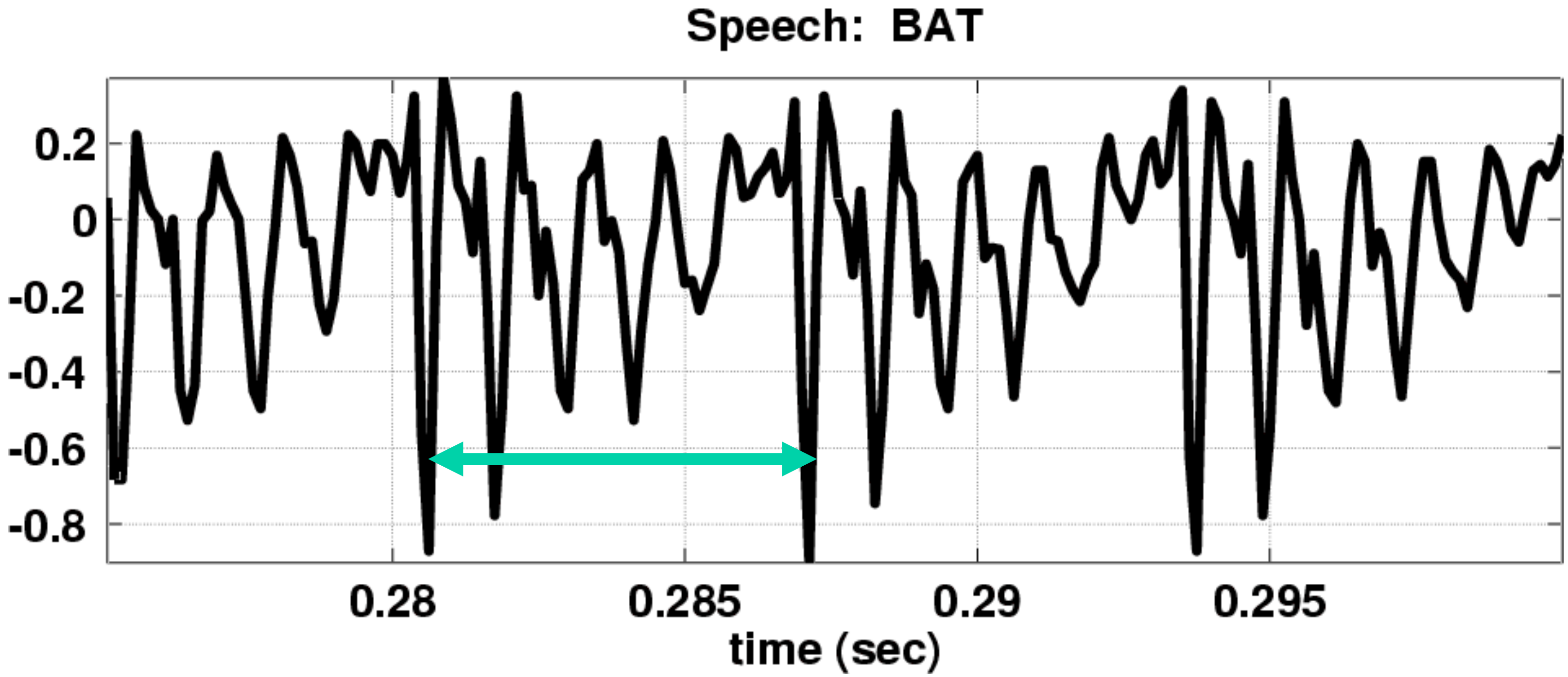


Beat Notes

Speech Signal: BAT



- Nearly **Periodic** in Vowel Region
  - Period is (Approximately) T = 0.0065 sec



Euler's Formula Reversed

- Solve for cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

$$\boxed{\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)}$$

INVERSE Euler's Formula

- Solve for cosine (or sine)

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

SPECTRUM Interpretation

- Cosine = sum of 2 complex exponentials:

$$A\cos(7t) = \frac{A}{2}e^{j7t} + \frac{A}{2}e^{-j7t}$$

One has a positive frequency
The other has negative freq.
Amplitude of each is half as big

SPECTRUM of SINE

- Sine = sum of 2 complex exponentials:

$$A\sin(7t) = \frac{A}{2j}e^{j7t} - \frac{A}{2j}e^{-j7t}$$

$$= \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

  - Positive freq. has phase = -0.5π
  - Negative freq. has phase = +0.5π

$$\frac{-1}{j} = j = e^{j0.5\pi}$$

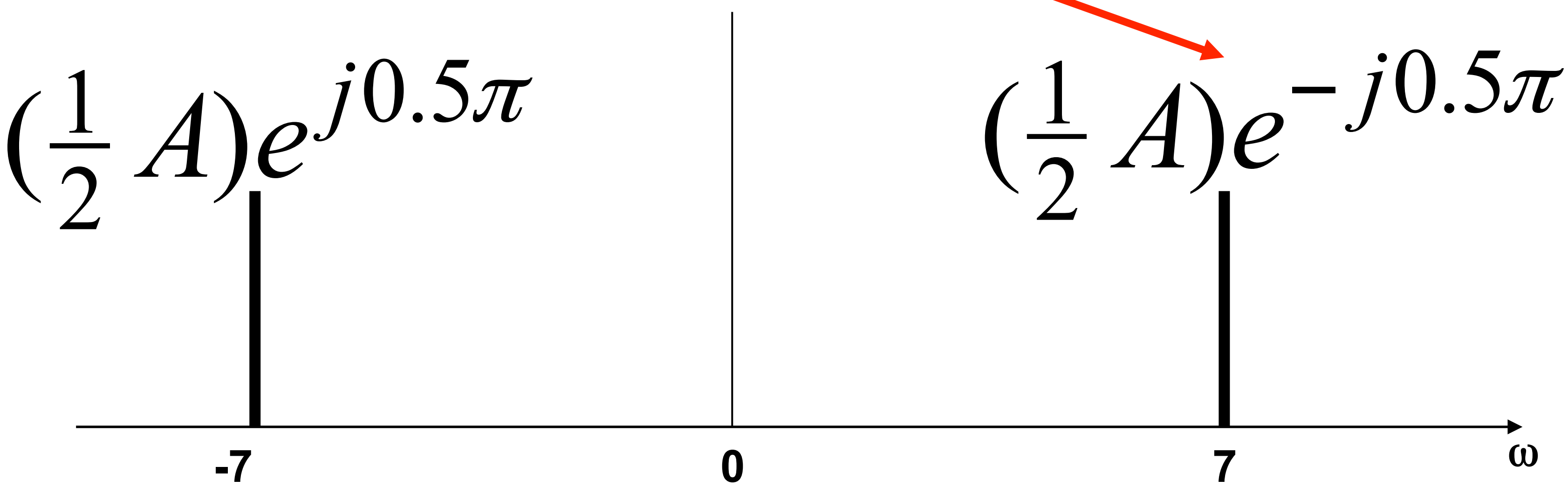
Negative Frequency

- Is negative frequency real?
- Doppler Radar provides an example
  - Police radar measures speed by using the Doppler shift principle
  - Let’s assume 400Hz ←→60 mph
  - +400Hz means towards the radar
  - -400Hz means away (opposite direction)
  - Think of a train whistle

Graphical Spectrum

EXAMPLE of SINE

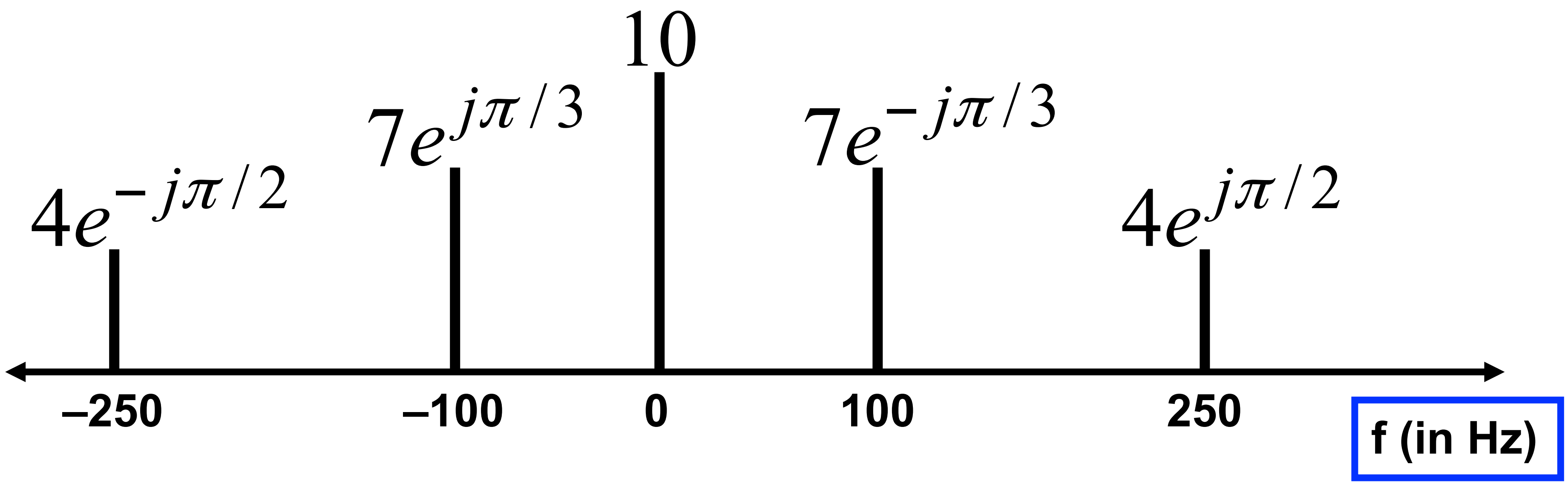
$$A\sin(7t) = \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$



AMPLITUDE, PHASE & FREQUENCY are shown

SPECTRUM2SINUSOID

- Add the spectrum components:



What is the formula for the signal x(t)?

Gather (A,ω,ϕ) information

- Frequencies:
  - -250 Hz
  - -100 Hz
  - **0** Hz
  - 100 Hz
  - 250 Hz

- Amplitude & Phase
  - 4 $\quad -\pi/2$
  - 7 $\quad +\pi/3$
  - 10 $\quad$ **0**
  - 7 $\quad -\pi/3$
  - 4 $\quad +\pi/2$

Note the **conjugate phase**

DC is another name for zero-freq component

DC component always has $\phi=0$ or π (for real ***x(t)***)

Add Spectrum Components-1

- **Frequencies:**
  - **-250 Hz**
  - **-100 Hz**
  - **0 Hz**
  - **100 Hz**
  - **250 Hz**

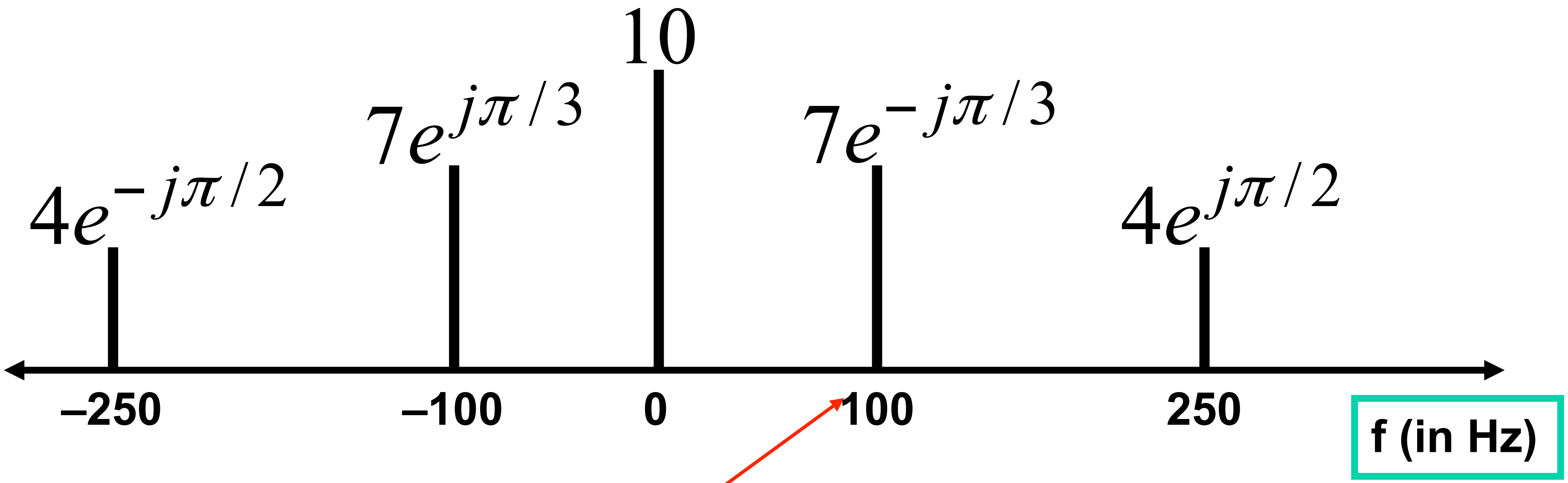
- **Amplitude & Phase**
  - **4 $-\pi/2$**
  - **7 $+\pi/3$**
  - **10 0**
  - **7 $-\pi/3$**
  - **4 $+\pi/2$**

$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Add Spectrum Components-2



$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Simplify Components

$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Use Euler's Formula to get REAL sinusoids:

$$A\cos(\omega t + \varphi) = \tfrac{1}{2}Ae^{+j\varphi}e^{j\omega t} + \tfrac{1}{2}Ae^{-j\varphi}e^{-j\omega t}$$

Final Answer

$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

<u>So, we get the general form:</u>

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

Summary: General Form

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\left\{X_k e^{j2\pi f_k t}\right\}$$

$$X_k = A_k e^{j\varphi_k}$$

$$\text{Frequency} = f_k$$

$$\Re e\{z\} = \tfrac{1}{2} z + \tfrac{1}{2} z^*$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{\tfrac{1}{2} X_k e^{j2\pi f_k t} + \tfrac{1}{2} X_k^* e^{-j2\pi f_k t}\right\}$$

Example: Synthetic Vowel

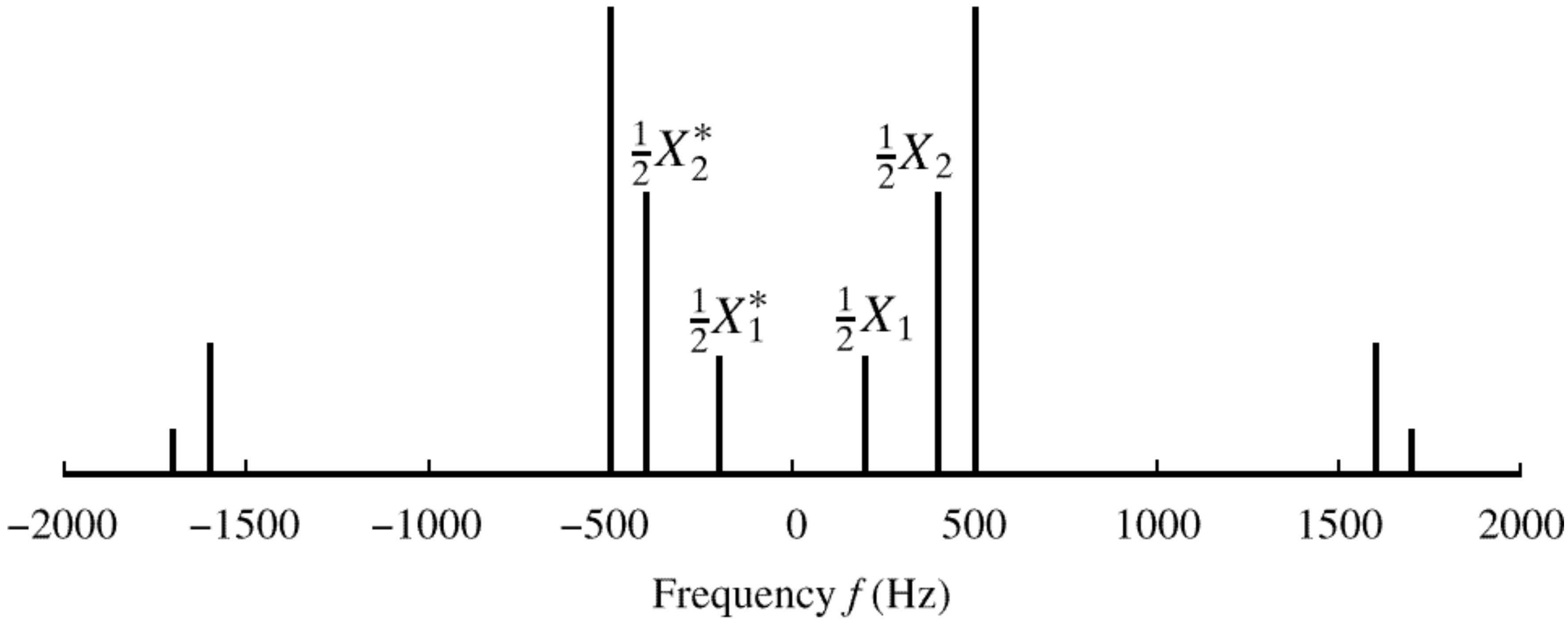
- Sum of 5 Frequency Components

f_k **(Hz)**	X_k	**Mag**	**Phase (rad)**
200	$(771 + j12202)$	12,226	1.508
400	$(-8865 + j28048)$	29,416	1.876
500	$(48001 - j8995)$	48,836	-0.185
1600	$(1657 - j13520)$	13,621	-1.449
1700	$4723 + j0$	4723	0

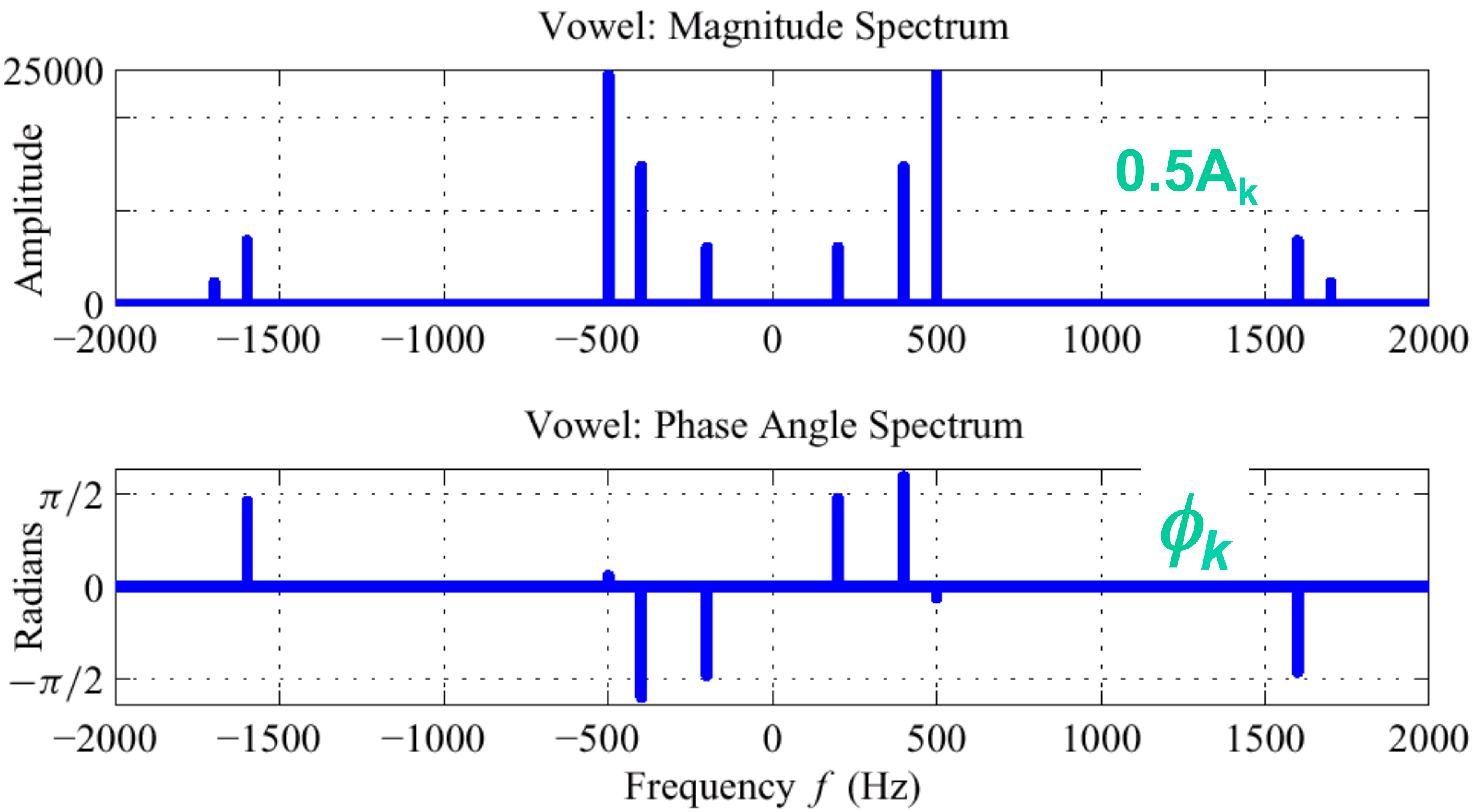
Table 3.1: Complex amplitudes for harmonic signal that approximates the vowel sound "ah".

SPECTRUM of VOWEL

- Note: Spectrum has $0.5X_k$ (except X_{DC})
- Conjugates in negative frequency



SPECTRUM of VOWEL (Polar Format)



Vowel Waveform
(sum of all 5 components)

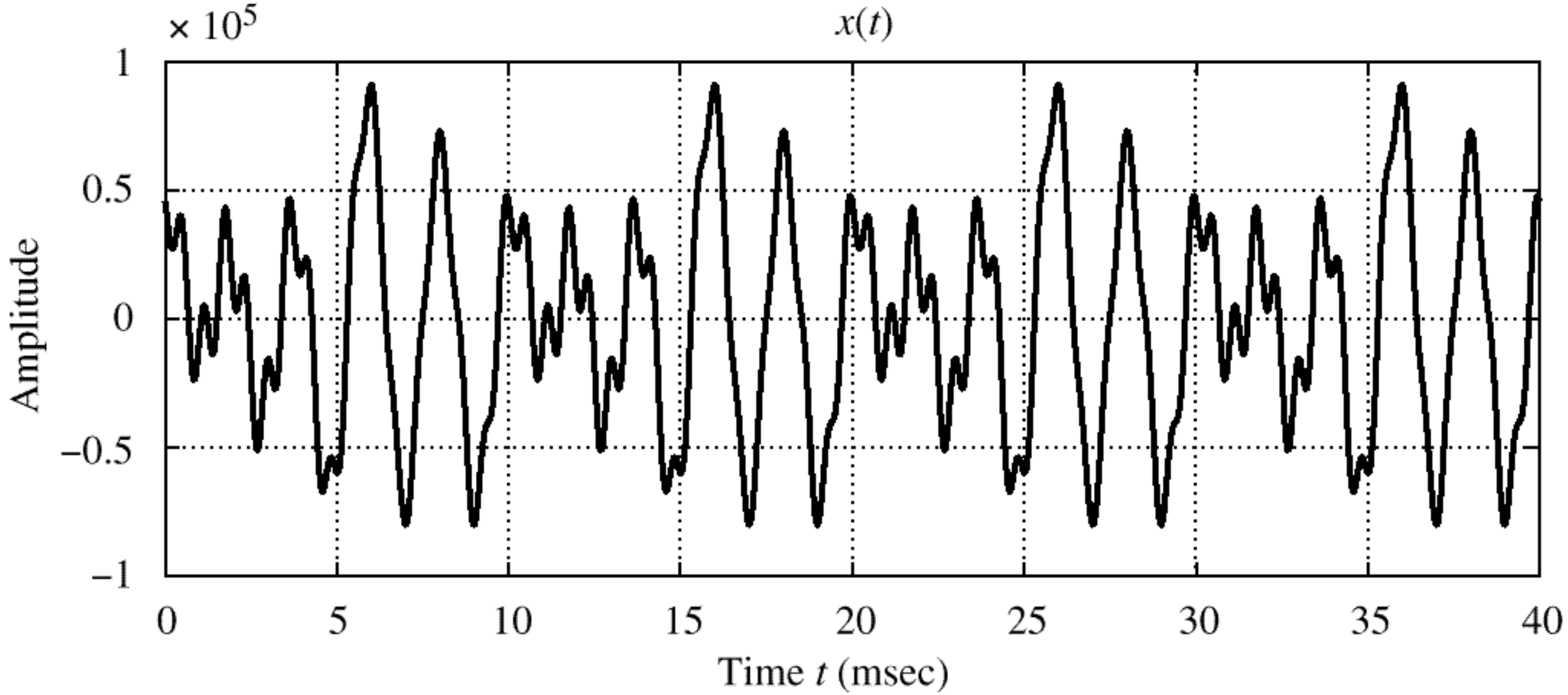
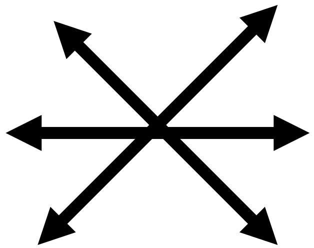


Figure 3.11 Sum of all of the terms in (3.3.4). Note that the period is 10 msec, which equals $1/f_0$.

Problem Solving Skills

- Math Formula
  - Sum of Cosines
  - Amp, Freq, Phase
- Recorded Signals
  - Speech
  - Music
  - No simple formula



- Plot & Sketches
  - S(t) versus t
  - Spectrum
- MATLAB
  - Numerical
  - Computation
  - Plotting list of numbers

Lecture Objectives

- Signals with HARMONIC Frequencies
  - Add Sinusoids with $f_k = kf_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

FREQUENCY can change vs. TIME

Chirps:

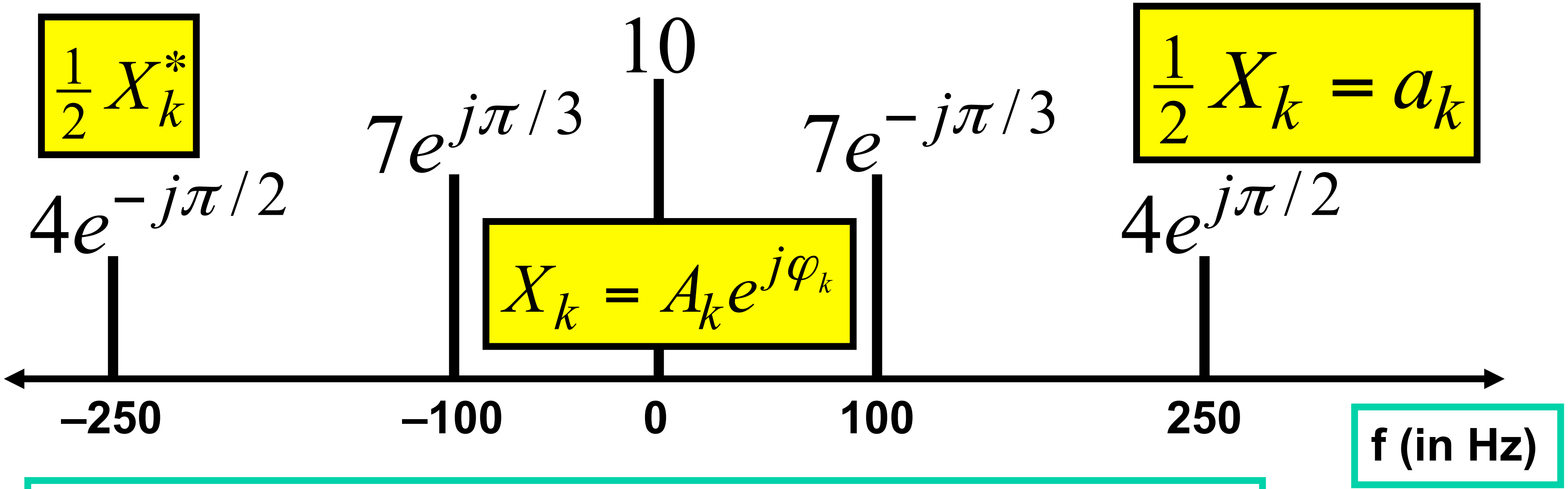
$$x(t) = \cos(\alpha t^2)$$

Introduce Spectrogram Visualization **(specgram.m)**

(plotspec.m)

Spectrum Diagram

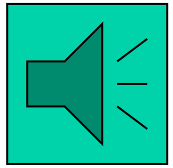
- Recall Complex Amplitude vs. Freq

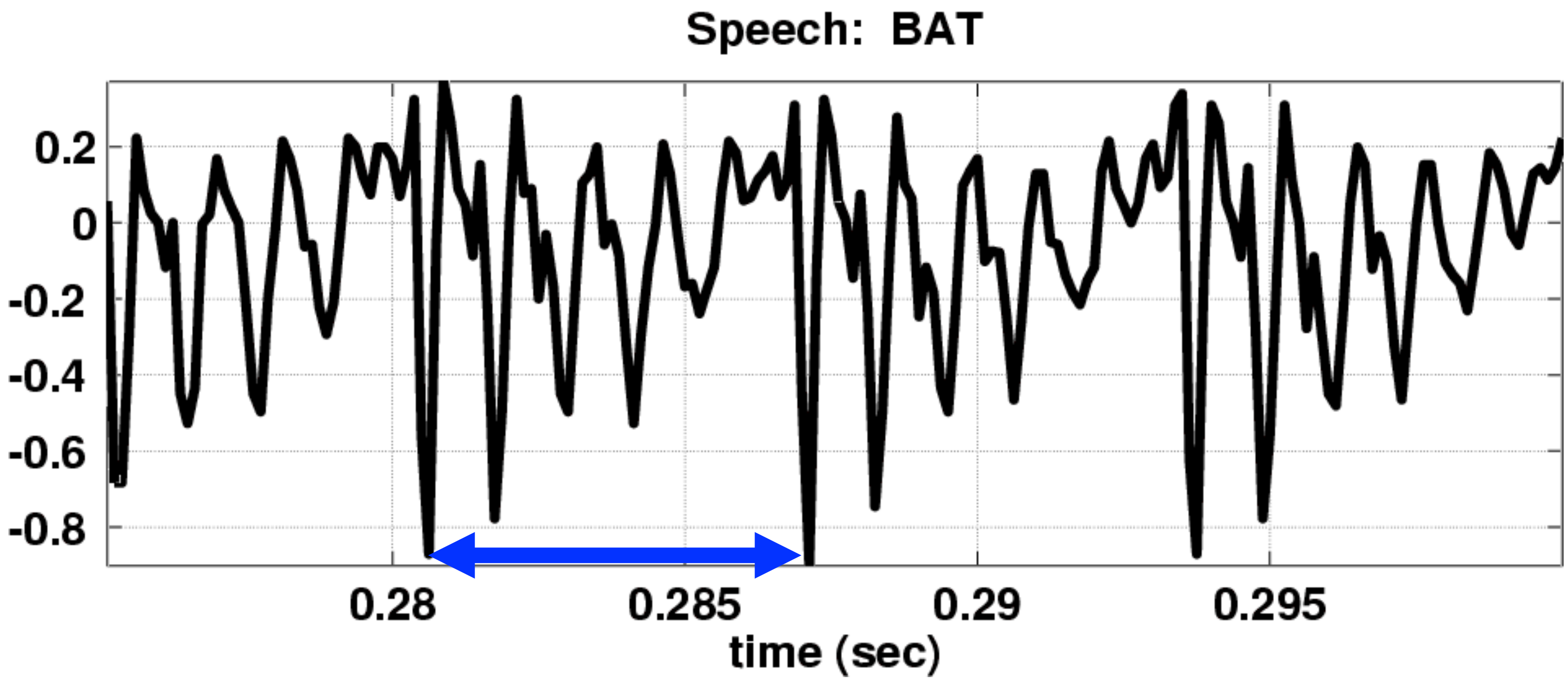


$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

Spectrum for Periodic Signals?

- Nearly Periodic in the Vowel Region
  - Period is (Approximately) T = 0.0065 sec





Periodic Signals

- Repeat every T secs
  - Definition

$$x(t) = x(t+T)$$

  - Example:

$$x(t) = \cos^2(3t)$$

$$T = ?$$

$$T = \frac{2\pi}{3}$$

$$T = \frac{\pi}{3}$$

  - Speech can be "quasi-periodic"

Period of Complex Exponentials

$$x(t) = e^{j\omega t}$$

$$x(t+T) = x(t)\ ?$$

Definition: Period is T

$$e^{j\omega(t+T)} = e^{j\omega t}$$

$$e^{j2\pi k} = 1$$

$$\Rightarrow e^{j\omega T} = 1 \quad \Rightarrow \omega T = 2\pi k$$

$$\omega = \frac{2\pi k}{T} = \left(\frac{2\pi}{T}\right)k = \omega_0 k$$

k = integer

Harmonic Signal Spectrum

Periodic signal can only have : $f_k = k f_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_0 = \frac{1}{T}$$

$$X_k = A_k e^{j\varphi_k}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t} \right\}$$

Define Fundamental Frequency

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

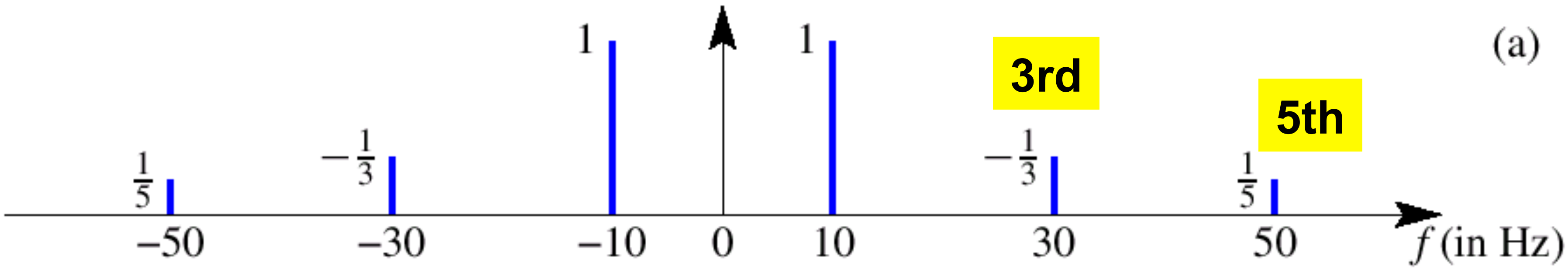
$$f_k = k f_0 \qquad (\omega_0 = 2\pi f_0)$$

$$f_0 = \frac{1}{T_0}$$

f_0 = fundamental Frequency (largest)

T_0 = fundamental Period (shortest)

Harmonic Signal (3 Freqs)

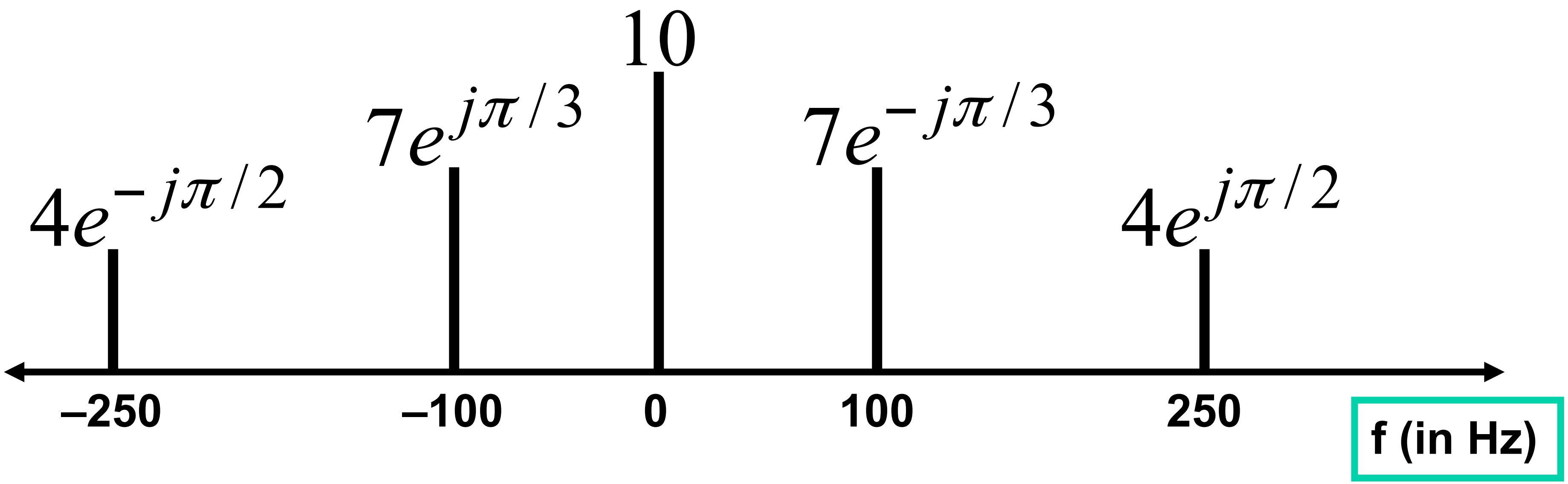


What is the fundamental frequency?

10 Hz

POP QUIZ: Fundamental Freq.

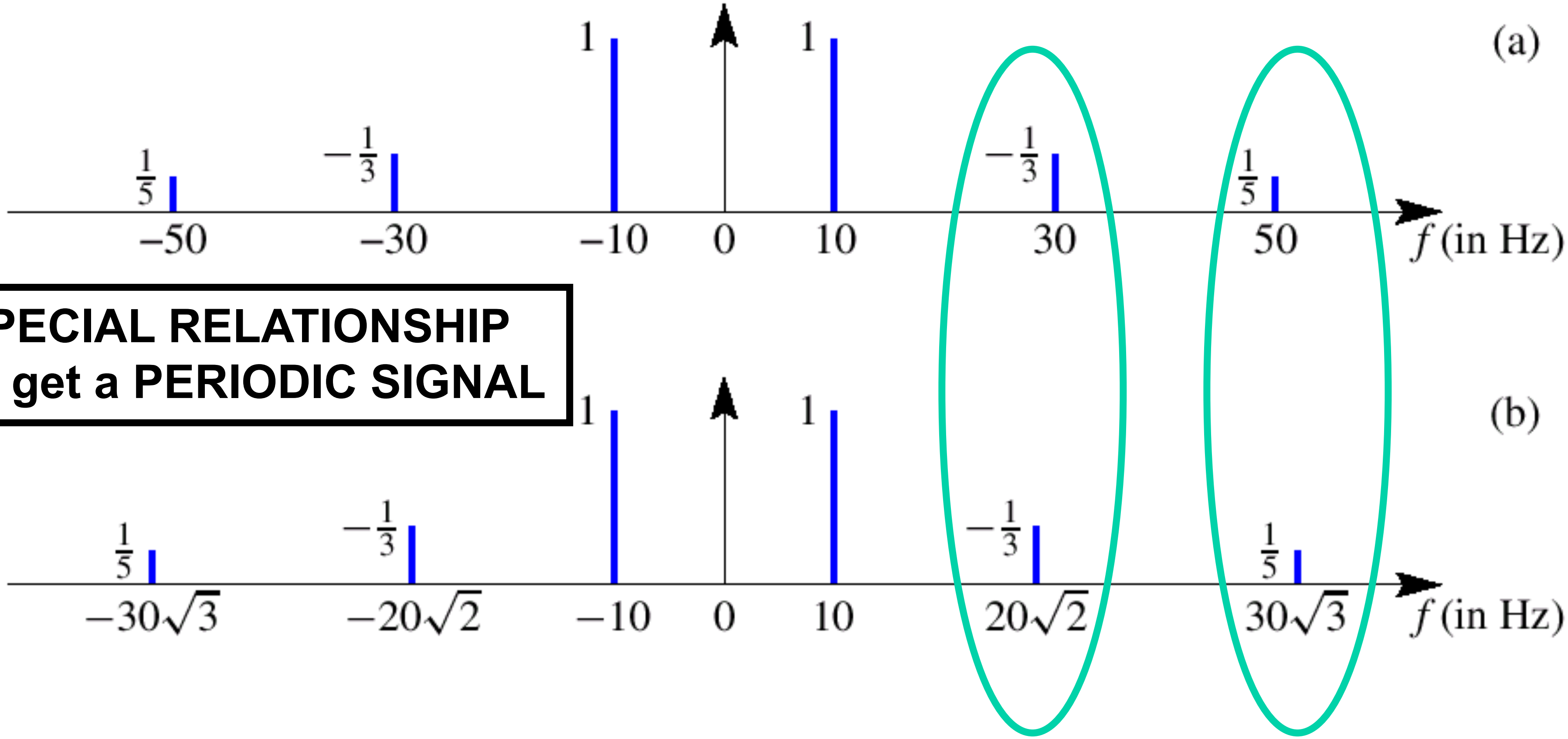
- Here's another spectrum:



What is the fundamental frequency?

100 Hz ? **50 Hz ?**

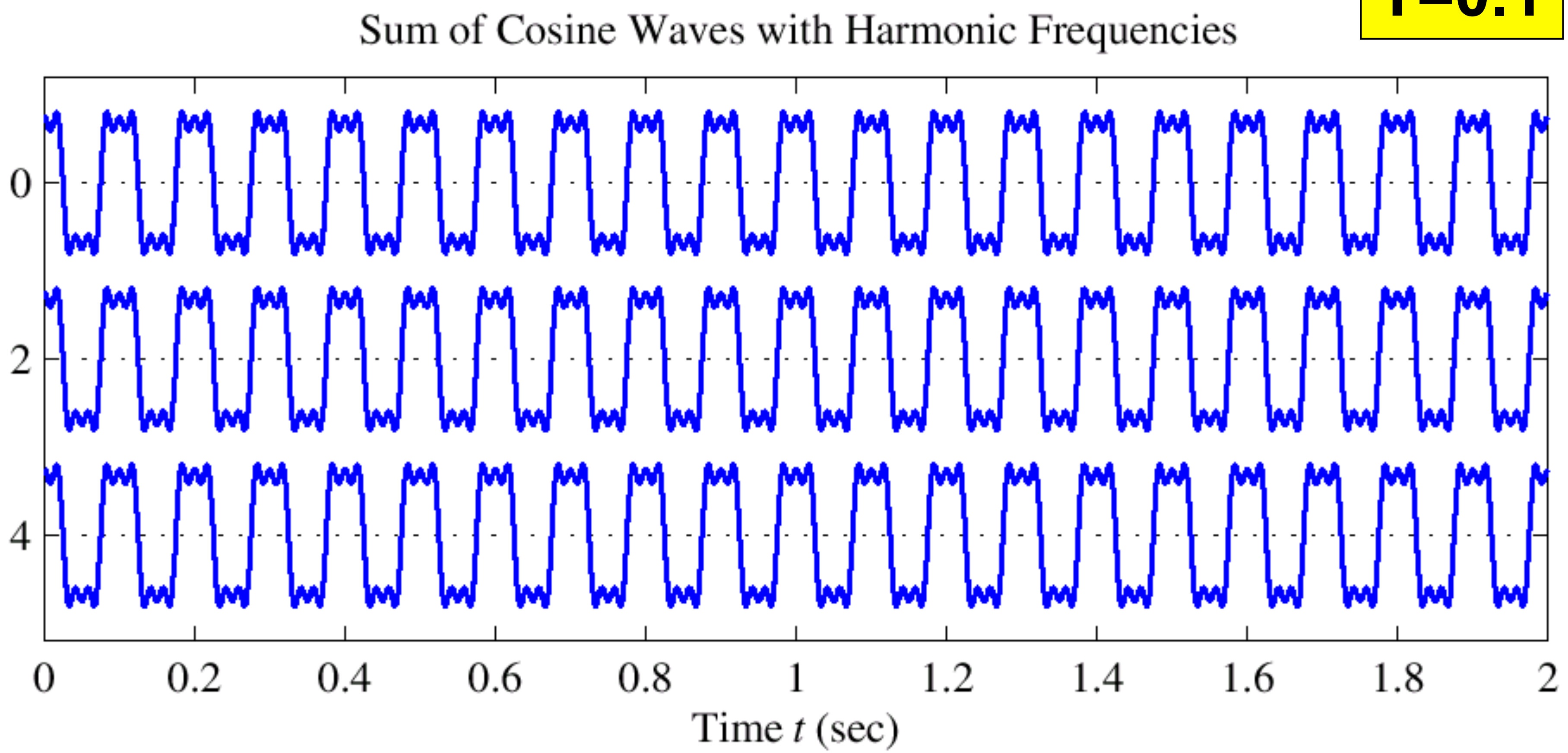
IRRATIONAL SPECTRUM



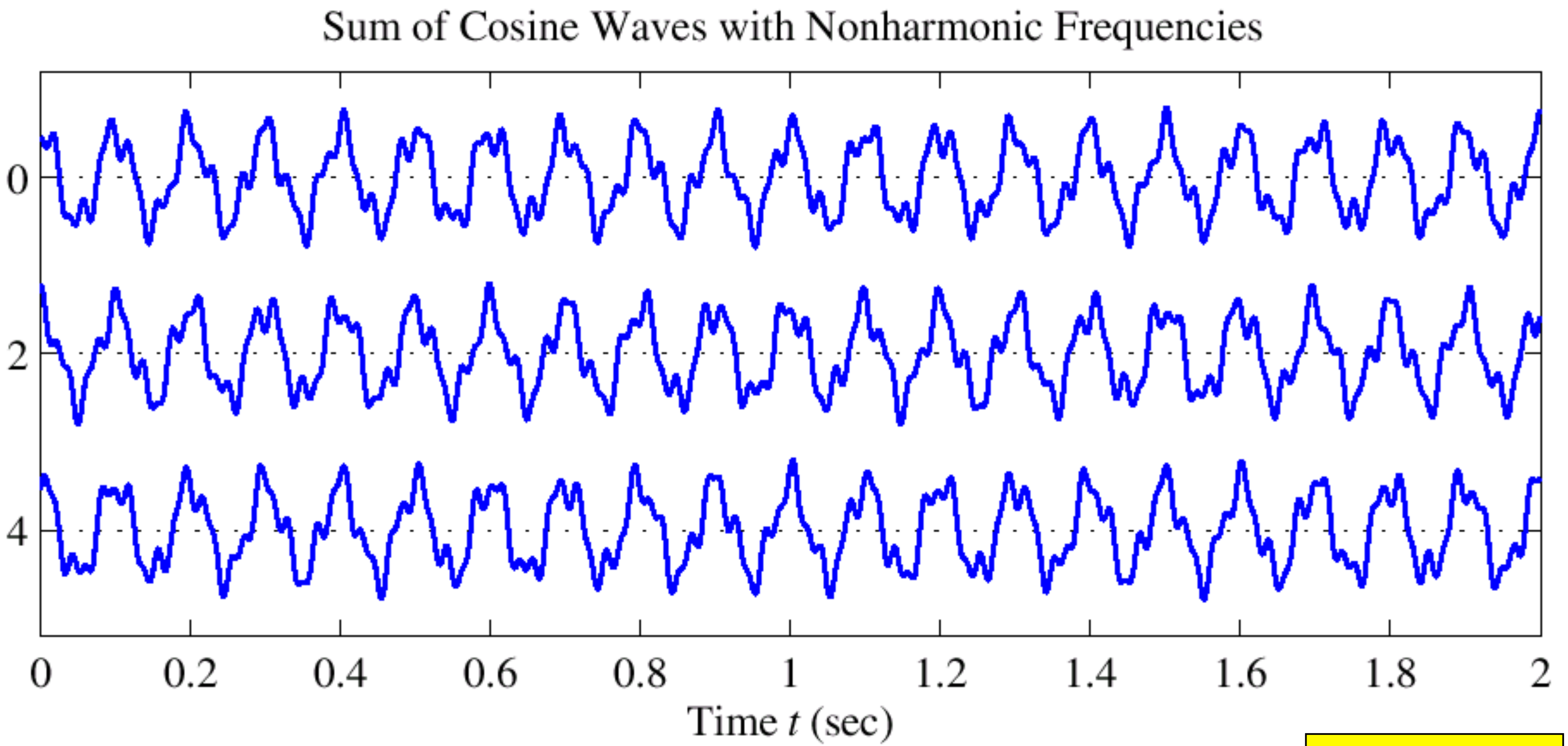
SPECIAL RELATIONSHIP to get a PERIODIC SIGNAL

Harmonic Signal (3 Freqs)

T=0.1



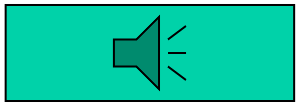
NON-Harmonic Signal

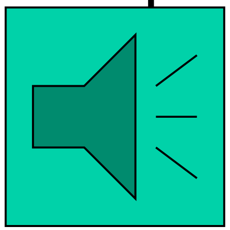


NOT PERIODIC

Frequency Analysis

- **<u>Now, a much HARDER problem</u>**
- Given a recording of a song, have the computer write the music





- Can a machine extract frequencies?
  - Yes, if we COMPUTE the spectrum for x(t)
    - During short intervals

Time-Varying FREQUENCIES Diagram

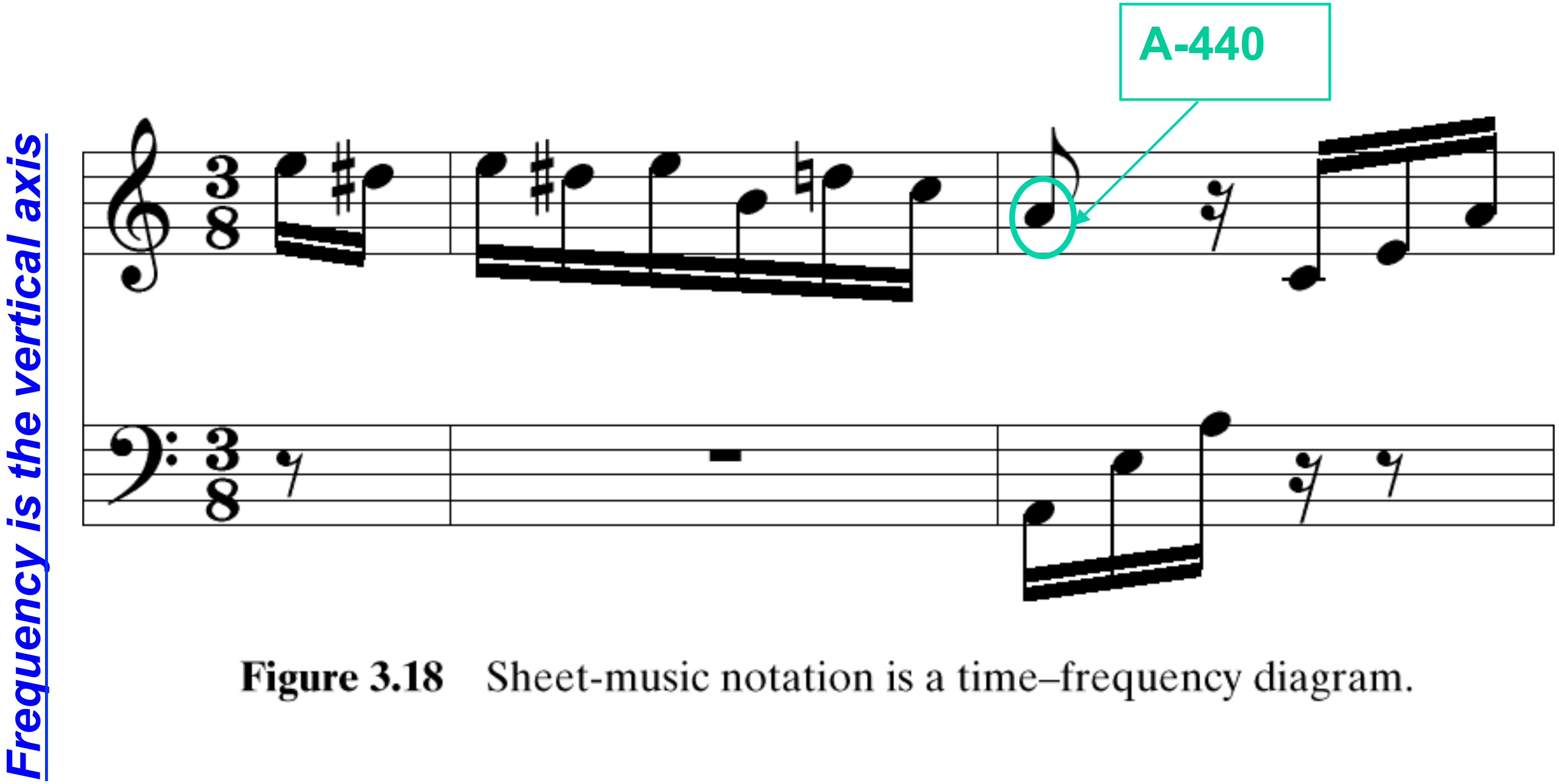
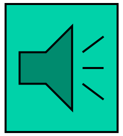


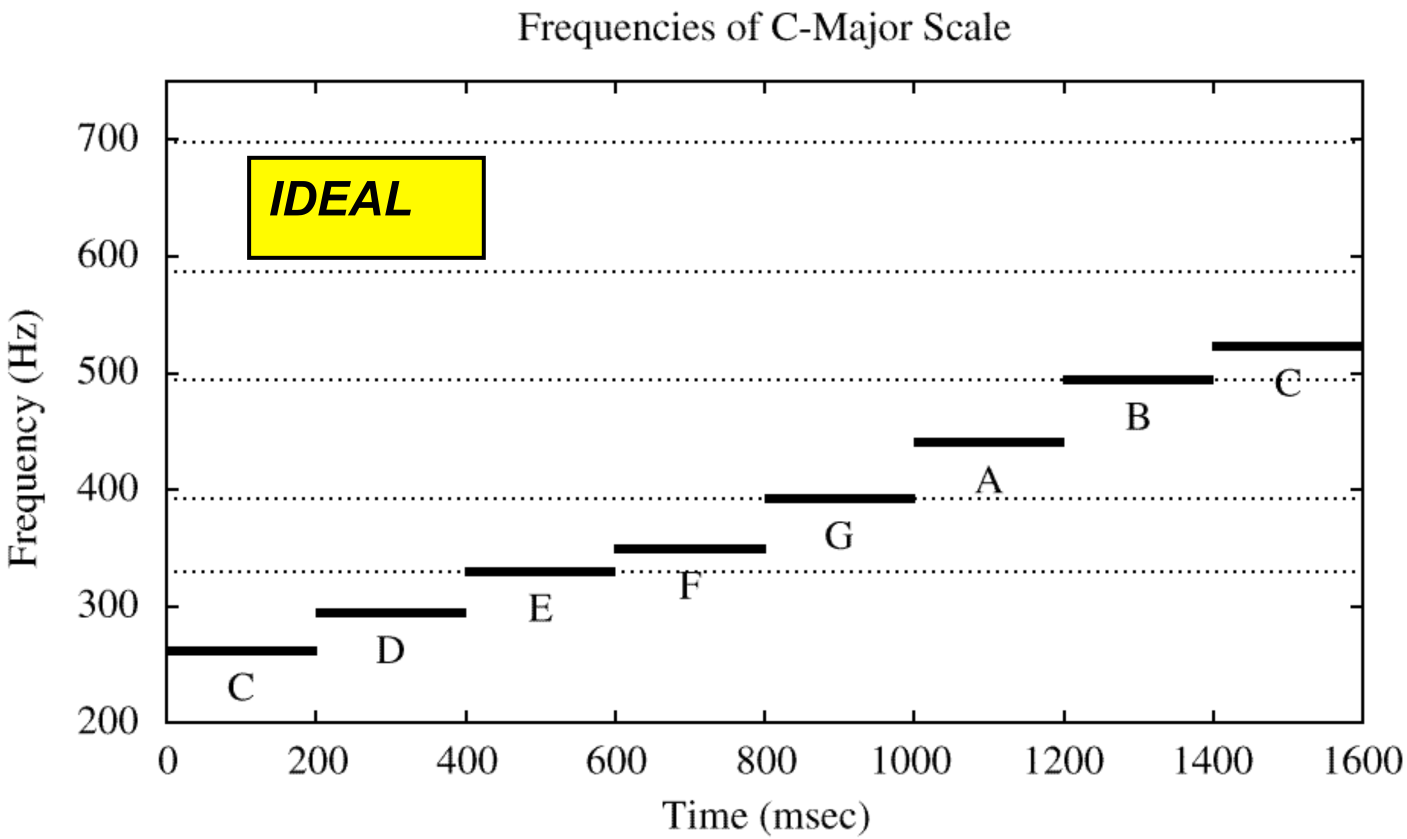
Figure 3.18 Sheet-music notation is a time–frequency diagram.

Time is the horizontal axis

A Simple Test Signal

- C-major SCALE: stepped frequencies
  - Frequency is constant for each note





R-rated: ADULTS ONLY

- SPECTROGRAM Tool
  - MATLAB function is `specgram.m`
  - SP-First has `plotspec.m` & `spectgr.m`
- **ANALYSIS** program
  - Takes x(t) as input &
  - Produces spectrum values X_k
  - Breaks x(t) into SHORT TIME SEGMENTS
    - Then uses the FFT (Fast Fourier Transform)

R-rated: ADULTS ONLY

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- **ANALYSIS** program
  - Takes x(t) as input &
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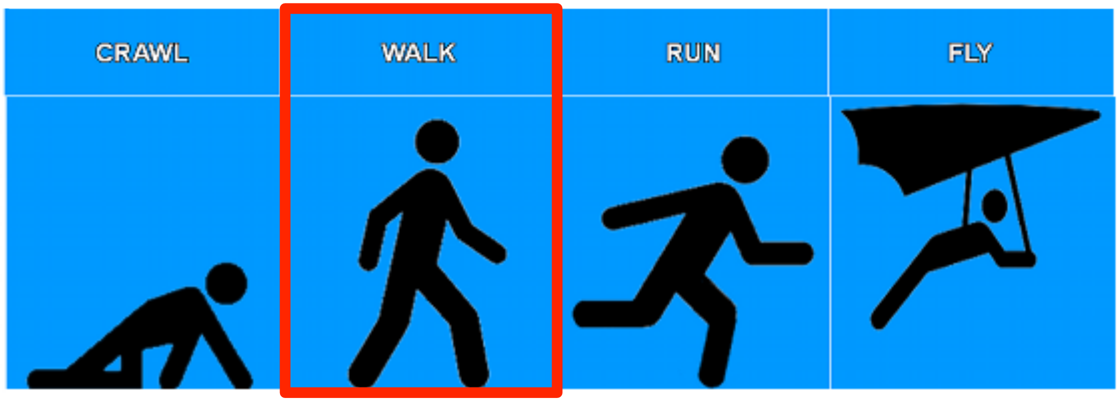
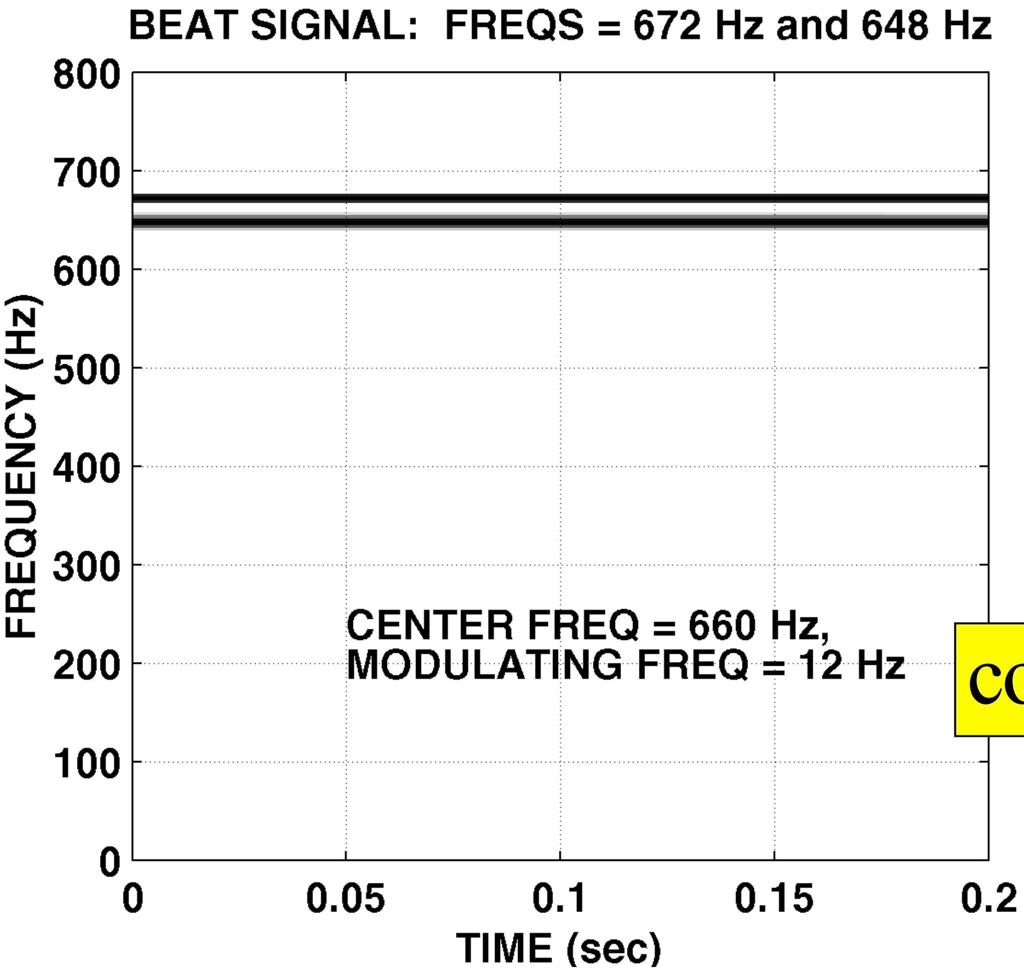
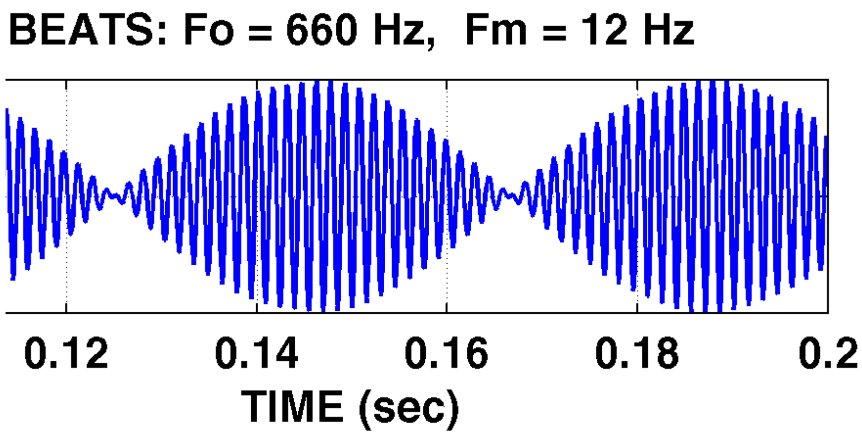
***CSI* Progress Level:**



Spectrogram Example

- Two **Constant** Frequencies: Beats





$$\cos(2\pi(660)t)\sin(2\pi(12)t)$$

AM Radio Signal

- Same as BEAT Notes

$$\cos(2\pi(660)t)\sin(2\pi(12)t)$$

$$\frac{1}{2}\left(e^{j2\pi(660)t}+e^{-j2\pi(660)t}\right)\frac{1}{2j}\left(e^{j2\pi(12)t}-e^{-j2\pi(12)t}\right)$$

$$\frac{1}{4j}\left(e^{j2\pi(672)t}-e^{-j2\pi(672)t}-e^{j2\pi(648)t}+e^{-j2\pi(648)t}\right)$$

$$\tfrac{1}{2}\cos(2\pi(672)t-\tfrac{\pi}{2})+\tfrac{1}{2}\cos(2\pi(648)t+\tfrac{\pi}{2})$$

Spectrum of AM (Beat)

- 4 complex exponentials in AM:

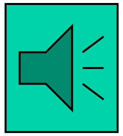
What is the fundamental frequency?

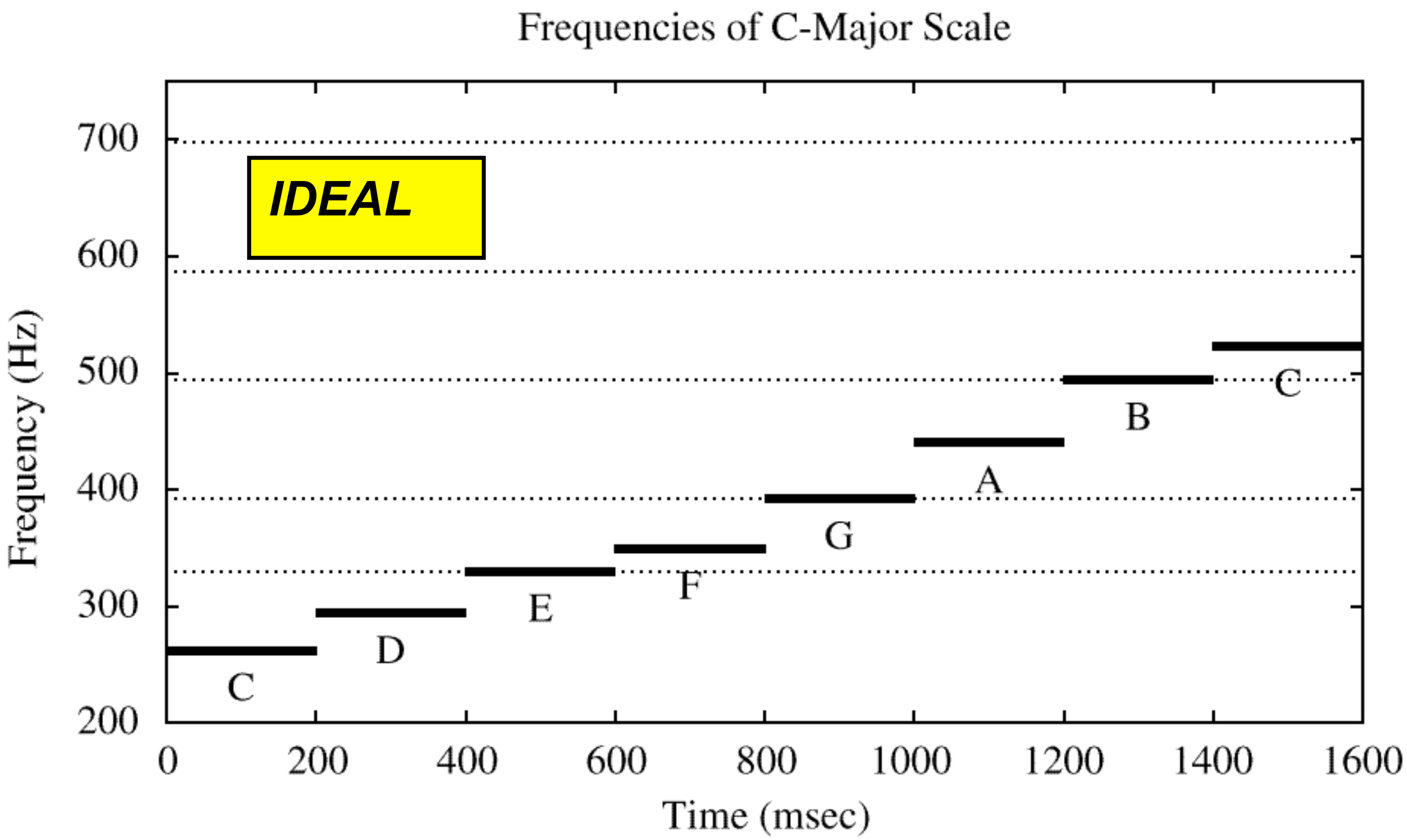
648 Hz ?

24 Hz ?

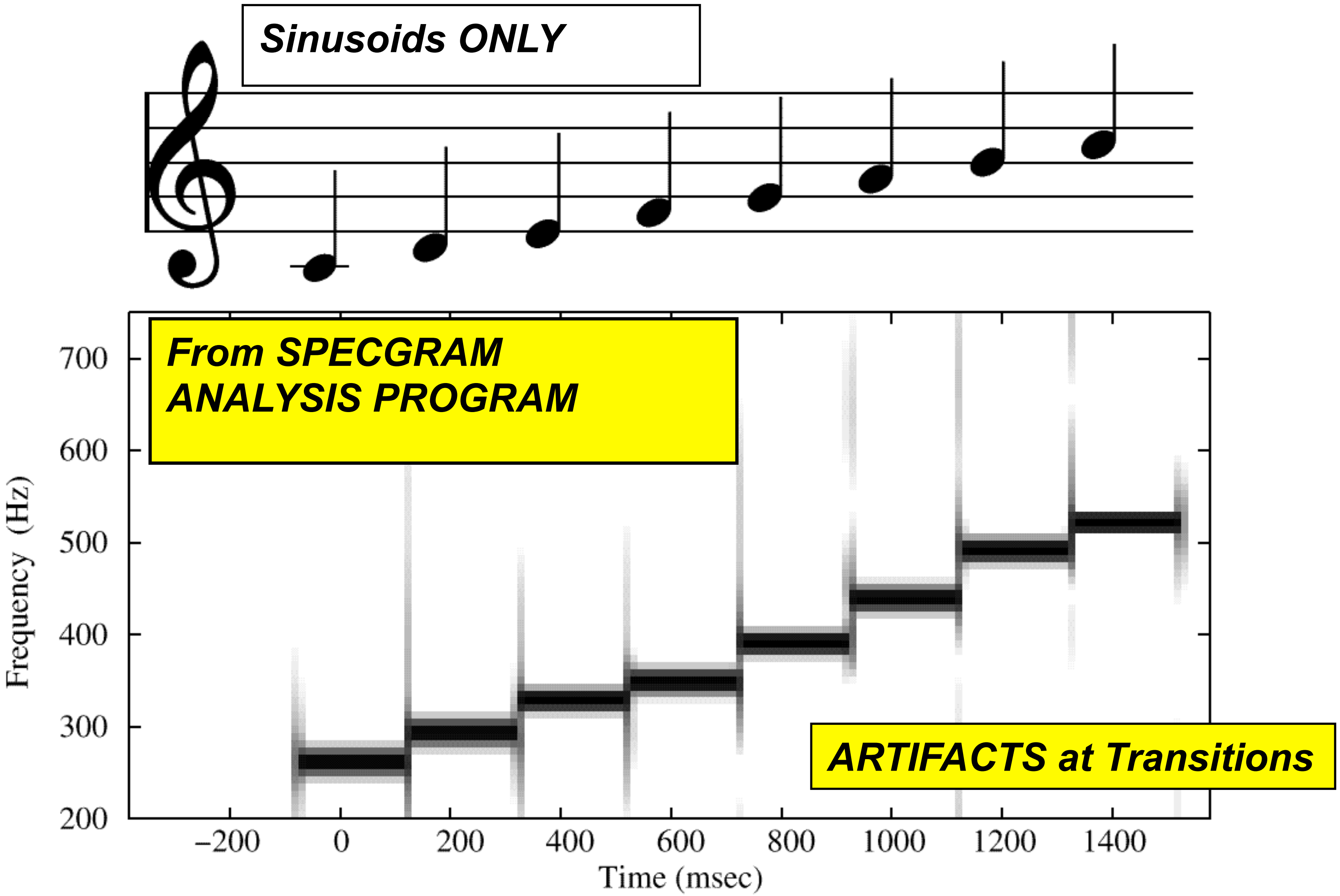
Stepped Frequencies

- C-major SCALE: successive sinusoids
  - Frequency is constant for each note

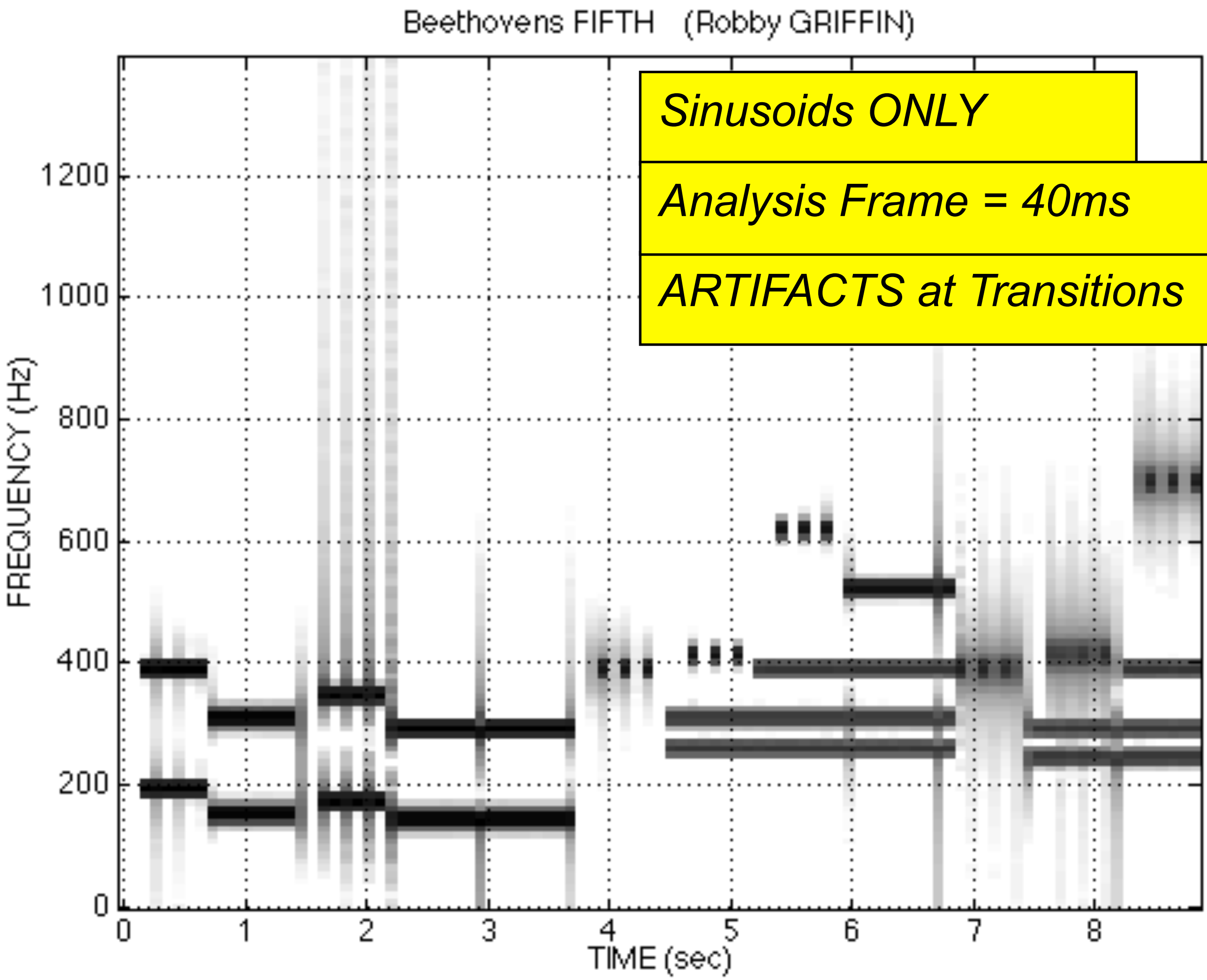




Spectrogram of C-Scale



Spectrogram of LAB SONG



Sinusoids ONLY

Analysis Frame = 40ms

ARTIFACTS at Transitions

Time-Varying Frequency

- Frequency can change vs. time
  - Continuously, not stepped
- **FREQUENCY MODULATION (FM)**

$$x(t) = \cos(2\pi f_c t + v(t))$$

VOICE

- CHIRP SIGNALS
  - Linear Frequency Modulation (LFM)

New Signal: Linear FM

- Called Chirp Signals (LFM)
  - Quadratic phase

QUADRATIC

$$x(t) = A\cos(\alpha t^2 + 2\pi f_0 t + \varphi)$$

- Freq will change LINEARLY vs. time
  - Example of Frequency Modulation (FM)
  - Define "instantaneous frequency"

Instantaneous Frequency

- Definition

$$x(t) = A\cos(\psi(t))$$
$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t)$$

Derivative of the "Angle"

- For Sinusoid:

$$x(t) = A\cos(2\pi f_0 t + \varphi)$$
$$\psi(t) = 2\pi f_0 t + \varphi$$
$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = 2\pi f_0$$

Makes sense

Instantaneous Frequency of the Chirp

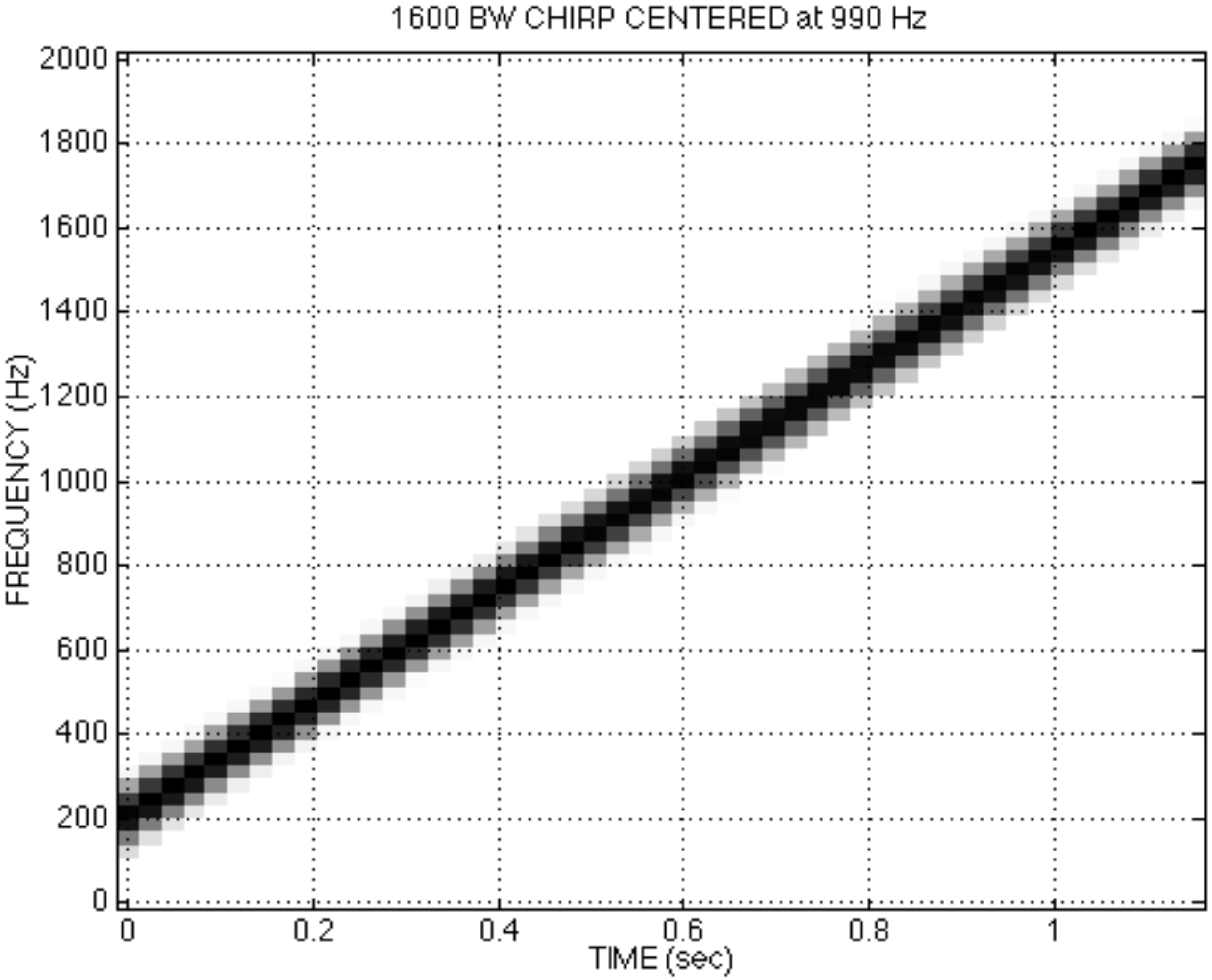
- Chirp Signals have Quadratic phase
- Freq will change LINEARLY vs. time

$$x(t) = A\cos(\alpha t^2 + \beta t + \varphi)$$

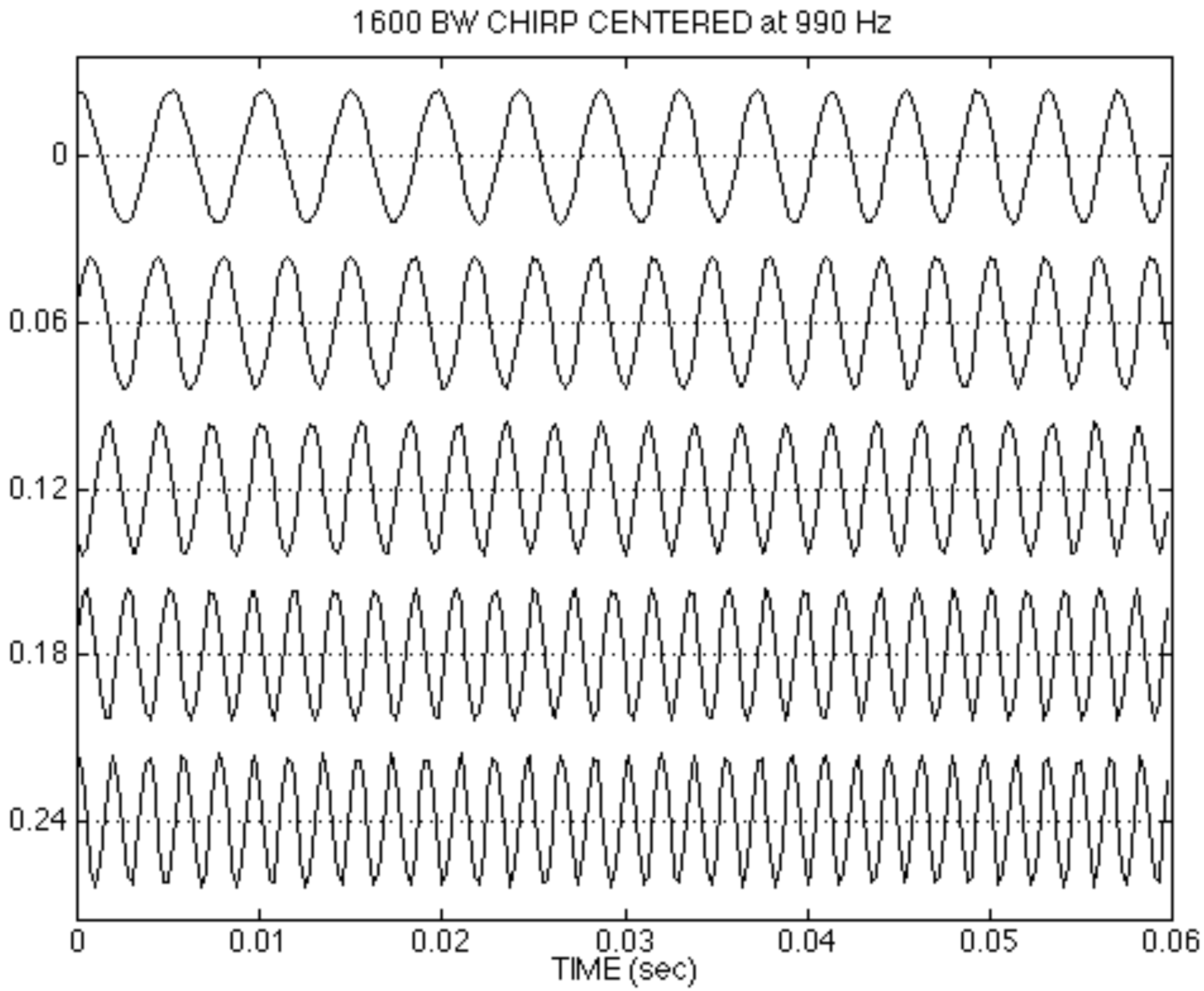
$$\Rightarrow \psi(t) = \alpha t^2 + \beta t + \varphi$$

$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = 2\alpha t + \beta$$

Chirp Spectrogram



Chirp Waveform



OTHER CHIRPS

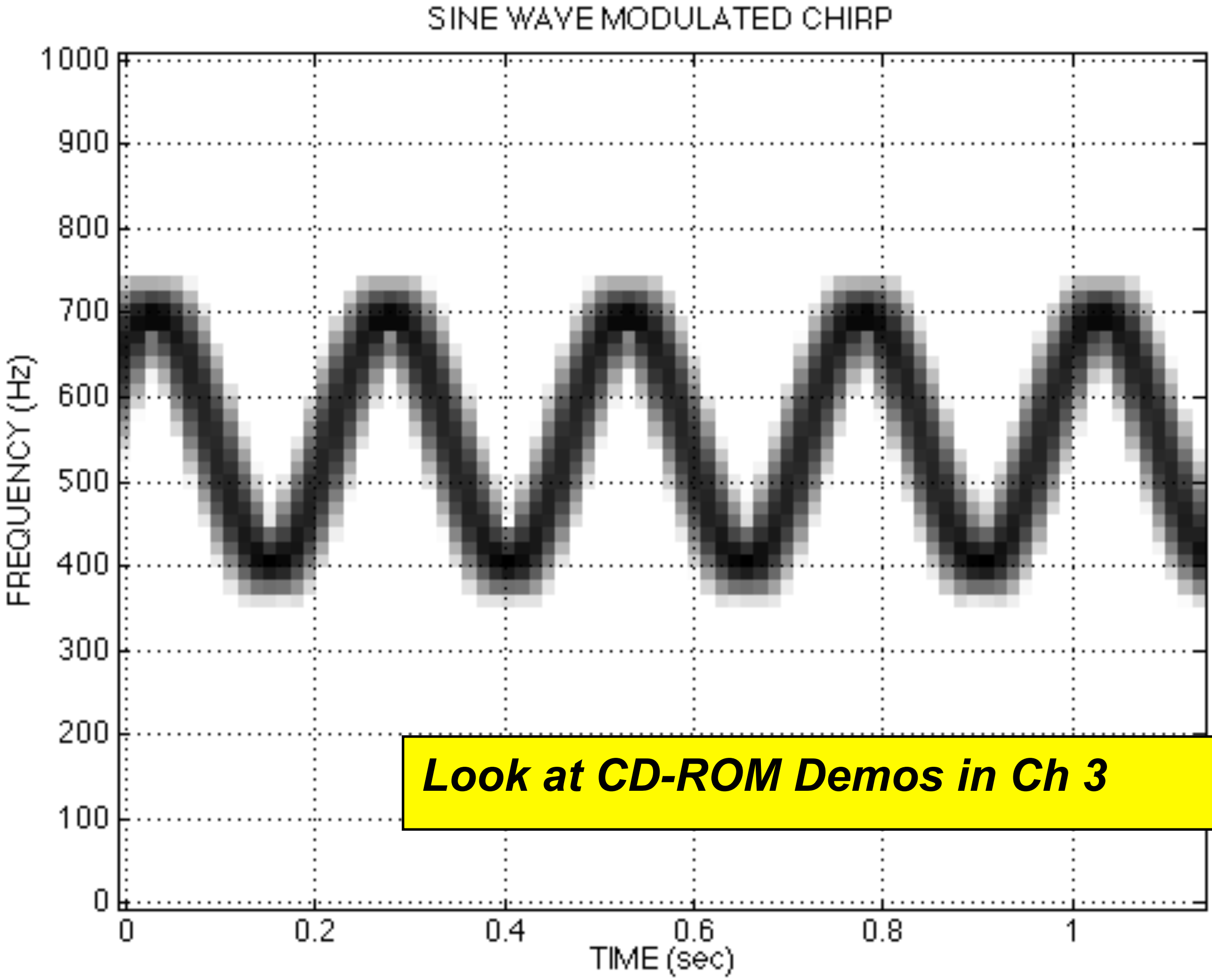
- $\psi(t)$ can be anything:

$$x(t) = A\cos(\alpha\cos(\beta t) + \varphi)$$

- $\psi(t)$ could be speech or music:
  - FM radio broadcast

$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = -\alpha\beta\sin(\beta t)$$

Sine-Wave
Frequency Modulation (FM)



Look at CD-ROM Demos in Ch 3

- Next week <> Section 3-4

Section 3-5

Section 3-6

Lab 2