



ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Circuits and Systems I

LECTURE #4

Fourier Series Analysis

lions@epfl

Prof. Dr. Volkan Cevher

LIONS/Laboratory for Information and Inference Systems

License Info for SPFirst Slides

- This work released under a [Creative Commons License](#) with the following terms:
- Attribution
 - The licensor permits others to copy, distribute, display, and perform the work. In return, licensees must give the original authors credit.
- Non-Commercial
 - The licensor permits others to copy, distribute, display, and perform the work. In return, licensees may not use the work for commercial purposes—unless they get the licensor's permission.
- Share Alike
 - The licensor permits others to distribute derivative works only under a license identical to the one that governs the licensor's work.
- [Full Text of the License](#)
- *This (hidden) page should be kept with the presentation*

Outline - Today

- Today <> Section 3-4
Section 3-5
Section 3-6
Lab 2
- Next week <> Same sections
(continued)
Lab 3

CSI
Progress
Level:



Lecture Objectives

- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi k / T_0)t} dt$$

- **ANALYSIS** via Fourier Series
 - For **PERIODIC** signals: $\mathbf{x}(t+T_0) = \mathbf{x}(t)$
 - Later: spectrum from the Fourier Series

HISTORY

- Jean Baptiste Joseph Fourier
 - 1807 thesis (memoir)
 - On the Propagation of Heat in Solid Bodies
 - Heat !
 - Napoleonic era
- <http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html>



Joseph Fourier

lived from 1768 to 1830

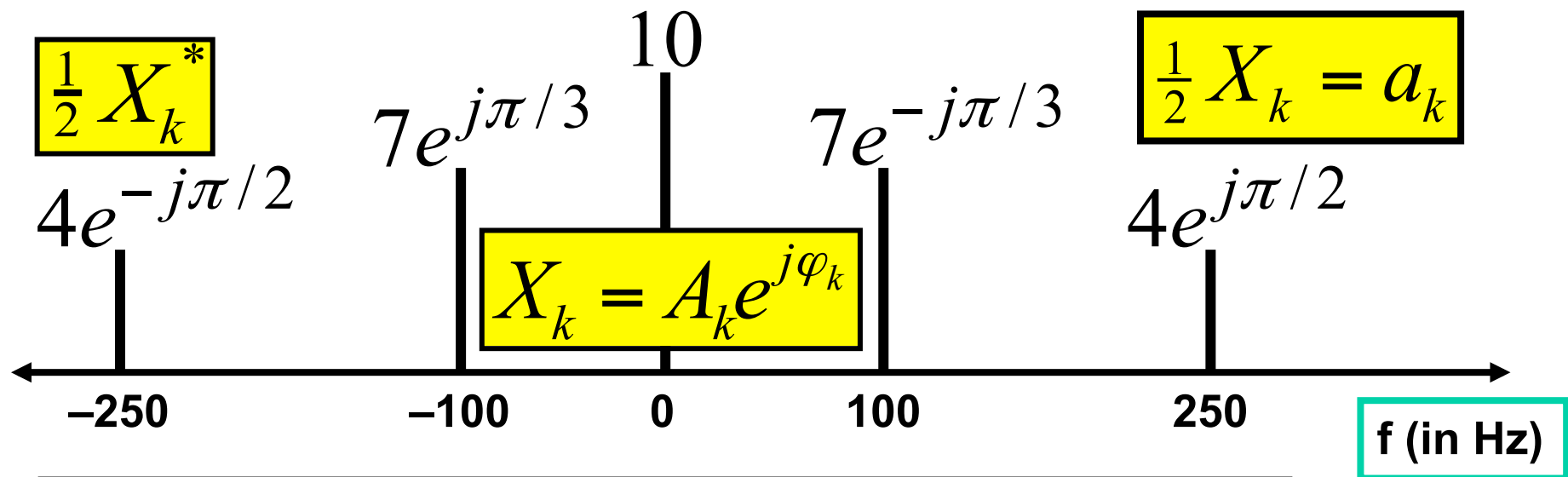
Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:

<http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html>

Spectrum Diagram

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^N \left\{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right\}$$

Harmonic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

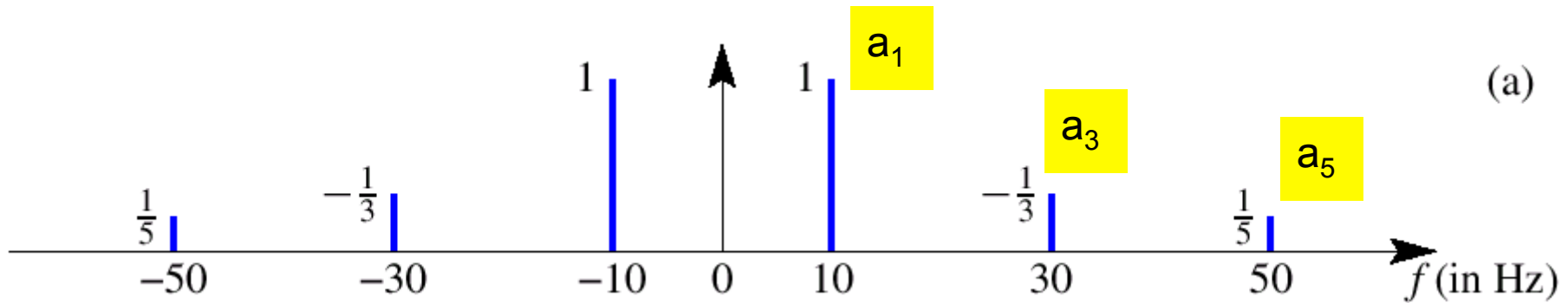
$$a_k = \frac{1}{2} X_k = \frac{1}{2} A_k e^{j\varphi_k}$$

$$x(t) = A_0 + \sum_{k=1}^N A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

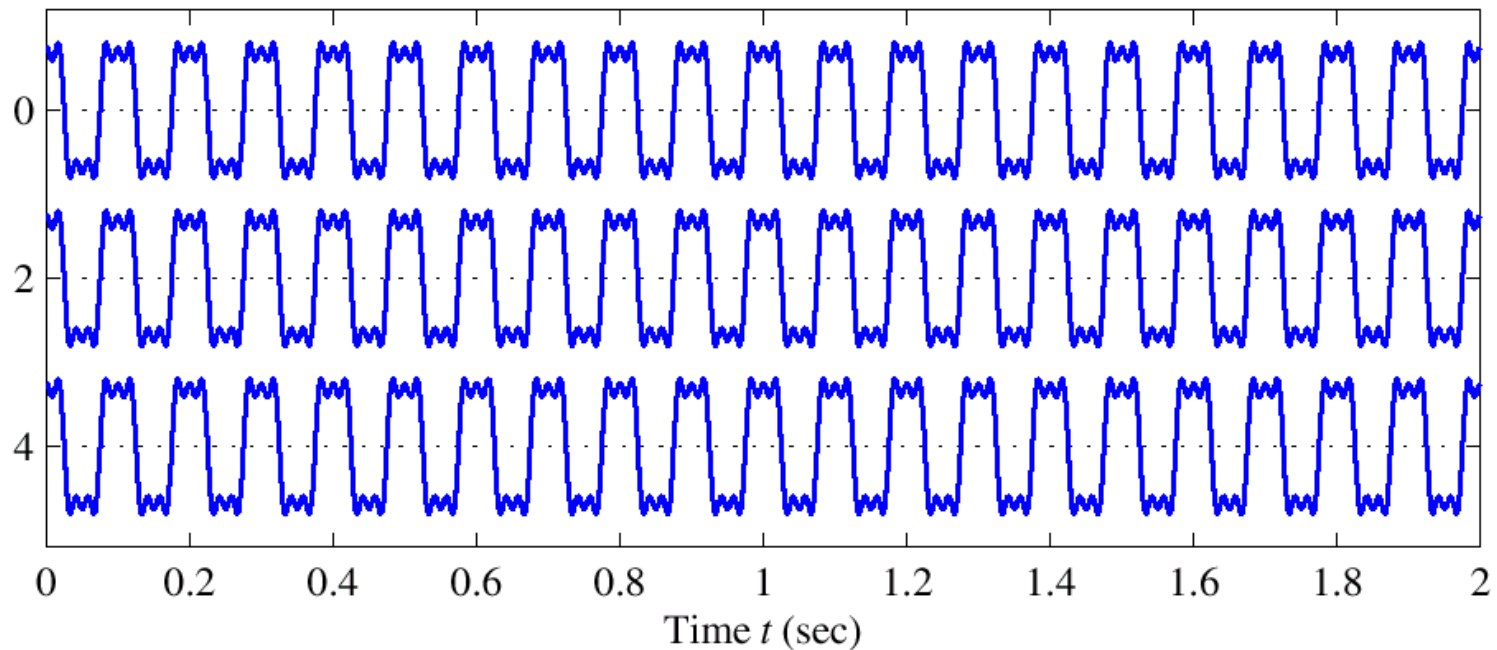
COMPLEX
AMPLITUDE

Harmonic Signal (3 Freqs)



Sum of Cosine Waves with Harmonic Frequencies

T = 0.1



Synthesis vs. Analysis

- SYNTHESIS

- **Easy**

- Given (ω_k, A_k, ϕ_k)
create $x(t)$

- Synthesis can be
HARD

- Synthesize Speech so that
it sounds good

- ANALYSIS

- **Hard**

- Given $x(t)$, extract
 (ω_k, A_k, ϕ_k)
 - How many?
 - Need algorithm for
computer

Strategy: $x(t) \rightarrow a_k$

- **ANALYSIS**

- Get representation from the signal
- Works for **PERIODIC** Signals

- Fourier Series

- Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

Integral Property of exp(j)

- INTEGRATE over ONE PERIOD

$$\begin{aligned}\int_0^{T_0} e^{-j(2\pi/T_0)mt} dt &= \frac{T_0}{-j2\pi m} e^{-j(2\pi/T_0)mt} \Big|_0^{T_0} \\ &= \frac{T_0}{-j2\pi m} (e^{-j2\pi m} - 1)\end{aligned}$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt} dt = 0 \quad m \neq 0 \quad \omega_0 = \frac{2\pi}{T_0}$$

Orthogonality of $\exp(j)$

- PRODUCT of $\exp(+j)$ and $\exp(-j)$

$$\frac{1}{T_0} \int_0^{T_0} e^{j(2\pi/T_0)\ell t} e^{-j(2\pi/T_0)kt} dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

$$\frac{1}{T_0} \int_0^{T_0} e^{j(2\pi/T_0)(\ell-k)t} dt$$

Isolate One FS Coefficient

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

$$\frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)\ell t} dt = \frac{1}{T_0} \int_0^{T_0} \left(\sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \right) e^{-j(2\pi/T_0)\ell t} dt$$

$$\frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)\ell t} dt = \sum_{k=-\infty}^{\infty} a_k \left(\frac{1}{T_0} \int_0^{T_0} e^{j(2\pi/T_0)kt} e^{-j(2\pi/T_0)\ell t} dt \right) = a_\ell$$

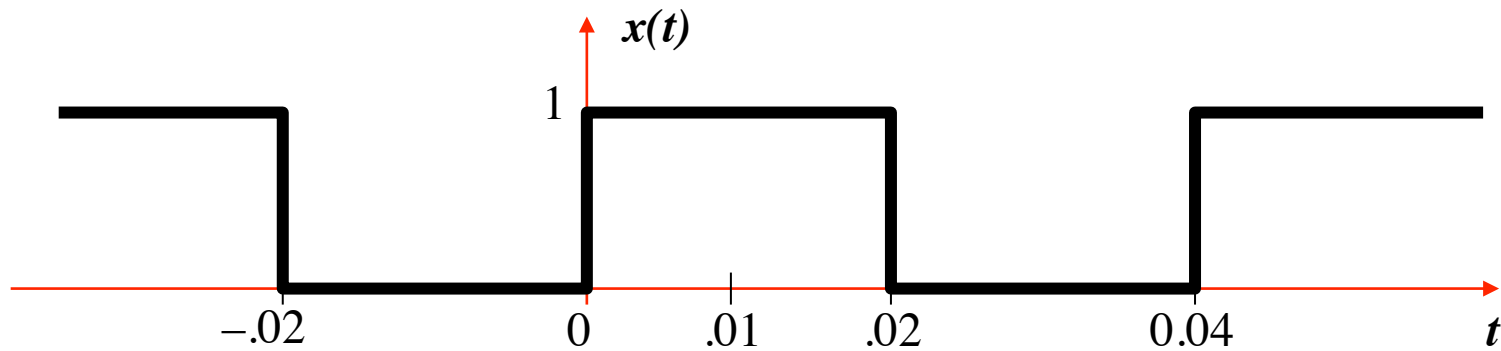
$$\Rightarrow a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

Integral is zero
except for $k = \ell$

Square Wave Example

$$x(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2} T_0 \\ 0 & \frac{1}{2} T_0 \leq t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



FS for a Square Wave $\{a_k\}$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k \neq 0)$$

$$a_k = \frac{1}{.04} \int_0^{.02} 1 e^{-j(2\pi/.04)kt} dt = \frac{1}{.04(-j2\pi k/.04)} e^{-j(2\pi/.04)kt} \Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)} (e^{-j(\pi)k} - 1) = \frac{1 - (-1)^k}{j2\pi k}$$

DC Coefficient: a_0

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k = 0)$$

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt = \frac{1}{T_0} (\text{Area})$$

$$a_0 = \frac{1}{.04} \int_0^{.02} 1 dt = \frac{1}{.04} (.02 - 0) = \frac{1}{2}$$

Fourier Coefficients a_k

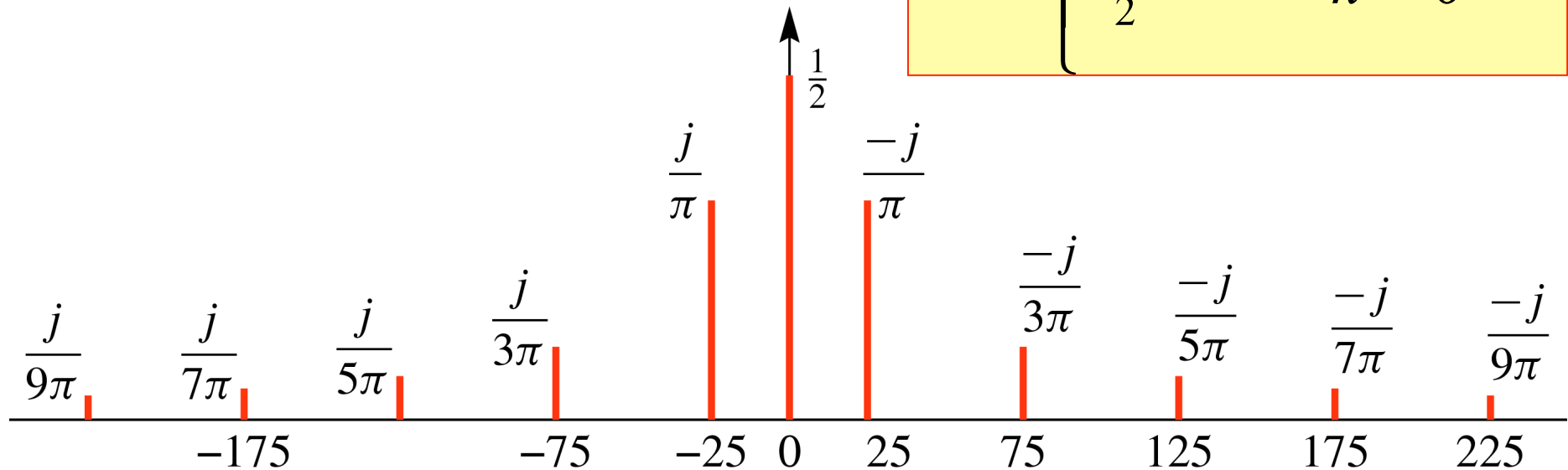
- a_k is a function of k
 - Complex Amplitude for k -th Harmonic
 - This one doesn't depend on the period, T_0

$$a_k = \frac{1 - (-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

$$\omega_0 = 2\pi / (0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$



Fourier Series Integral

- HOW do you determine a_k from $x(t)$?

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

Fundamental Frequency $f_0 = 1/T_0$

$a_{-k} = a_k^*$ when $x(t)$ is real

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt \quad (\text{DC component})$$



That's all Folks!

- Next week <>

Section 3-4
Section 3-5
Section 3-6
Lab 3

<>
LAB TIME NOW!