ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Circuits and Systems I

LECTURE #6

Sampling and Aliasing

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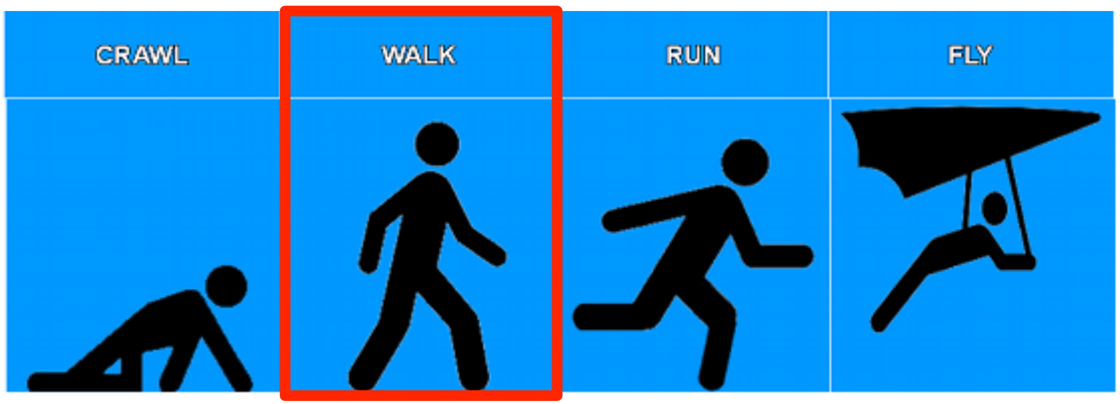
Outline - Today

- Today <> Section 4-1
  Section 4-2

- Next week <> Section 4-4
  Section 4-5
  Lab 4
  READ

CSI **Progress Level:**



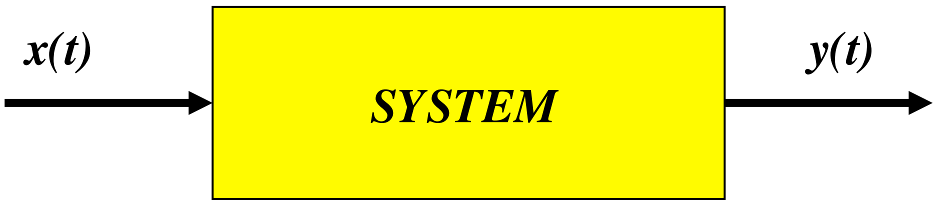
Lecture Objectives

- SAMPLING can cause ALIASING
  - **Sampling Theorem**
  - Sampling Rate > 2(Highest Frequency)
- Spectrum for digital signals, x[n]
  - Normalized Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi\ell$$

ALIASING

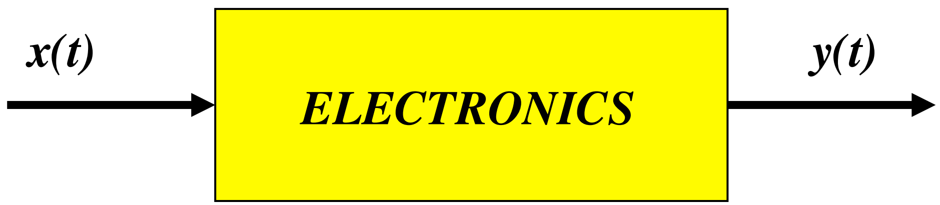
SYSTEMS Process Signals



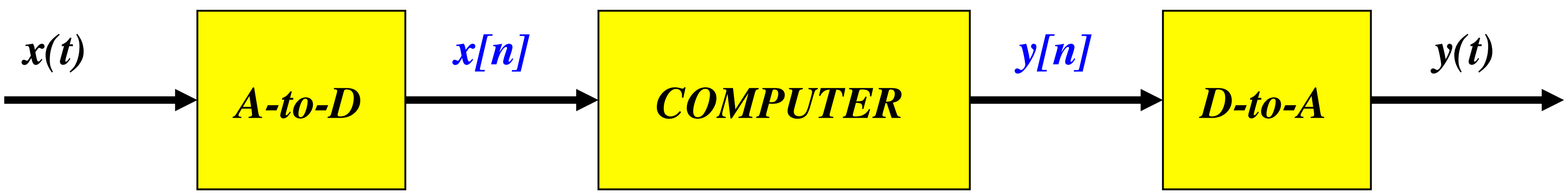
- PROCESSING GOALS:
  - Change x(t) into y(t)
    - For example, more BASS
  - Improve x(t), e.g., image deblurring
  - Extract Information from x(t)

System Implementation

- ANALOG/ELECTRONIC:
  - Circuits: resistors, capacitors, op-amps

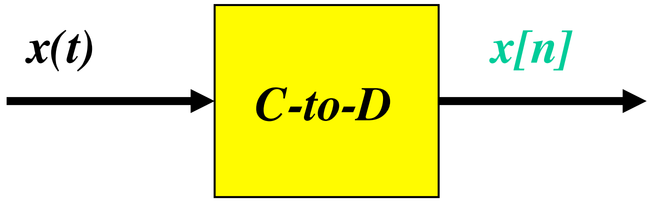


- DIGITAL/MICROPROCESSOR
  - Convert x(t) to numbers stored in memory



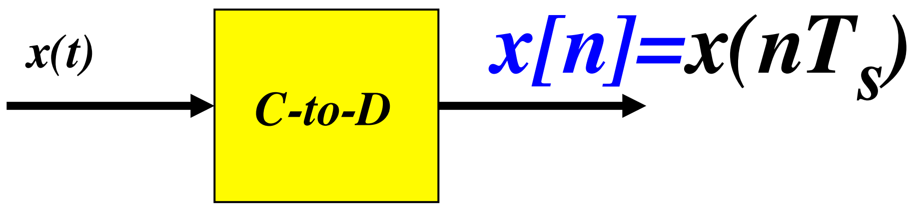
Sampling x(t)

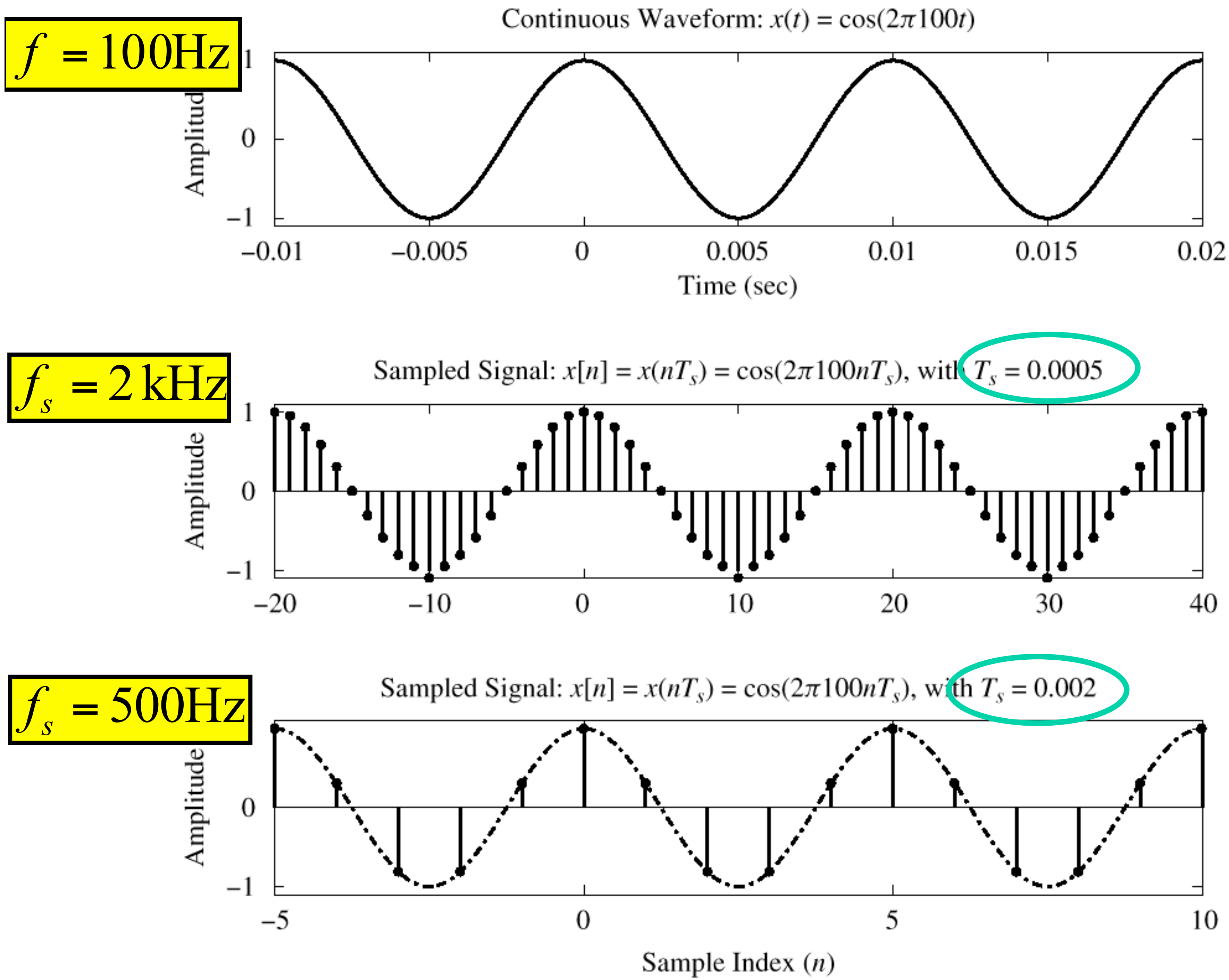
- SAMPLING PROCESS
  - Convert x(t) to numbers x[n]
  - "n" is an integer; x[n] is a sequence of values
  - Think of "n" as the storage address in memory
- UNIFORM SAMPLING at $t = nT_s$
  - IDEAL: $x[n] = x(nT_s)$



Sampling Rate, f_s

- SAMPLING RATE (f_s)
  - $f_s = 1/T_s$
    - NUMBER of SAMPLES PER SECOND
  - T_s = 125 microsec → f_s = 8000 samples/sec
    - UNITS ARE HERTZ: 8000 Hz
- UNIFORM SAMPLING at $t = nT_s = n/f_s$
  - IDEAL: $x[n] = x(nT_s) = x(n/f_s)$





f = 100Hz
Continuous Waveform: x(t) = cos(2π100t)
Amplitude
1
0
−1
−0.01
−0.005
0
0.005
0.01
0.015
0.02
Time (sec)
fs = 2 kHz
Sampled Signal: x[n] = x(nTs) = cos(2π100nTs), with Ts = 0.0005
Amplitude
1
0
−1
−20
−10
0
10
20
30
40
fs = 500Hz
Sampled Signal: x[n] = x(nTs) = cos(2π100nTs), with Ts = 0.002
Amplitude
0
−1
−5
0
5
10
Sample Index (n)

Sampling Theorem

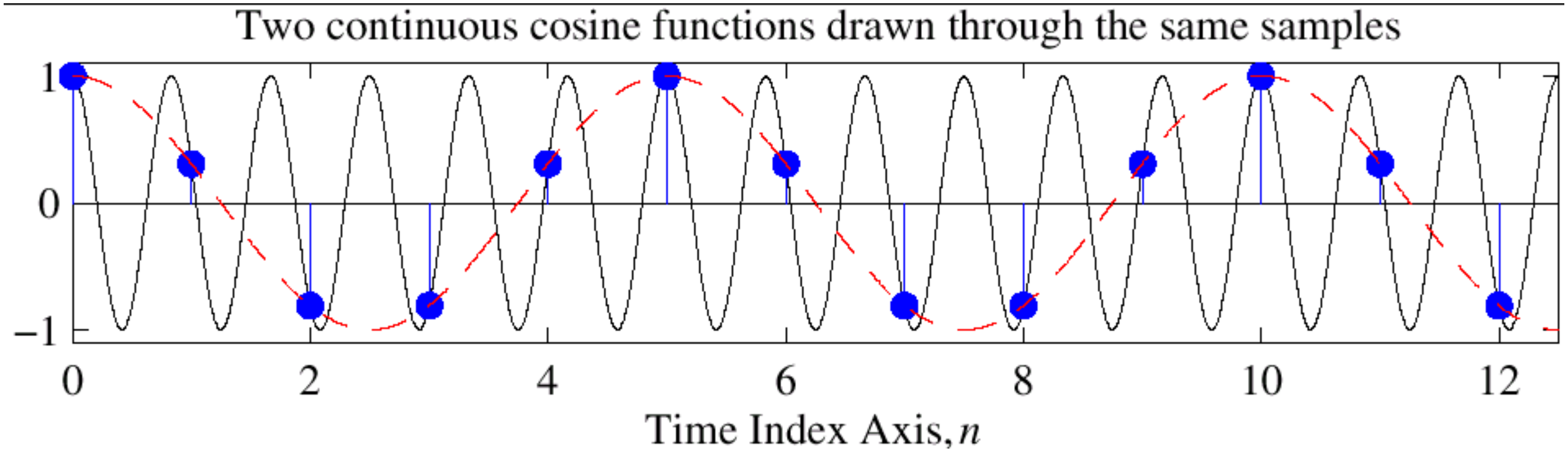
- HOW OFTEN ?
  - DEPENDS on FREQUENCY of SINUSOID
  - ANSWERED by SHANNON/NYQUIST Theorem
  - ALSO DEPENDS on "**RECONSTRUCTION**"

Shannon Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than $f_{\max}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\max}$.

Reconstruction? Which One?

Given the samples, draw a sinusoid through the values



$$x[n] = \cos(0.4\pi n)$$

When n is an integer

$$\cos(0.4\pi n) = \cos(2.4\pi n)$$

Storing Digital Sound

- $x[n]$ is a SAMPLED SINUSOID
  - A list of numbers stored in memory
- EXAMPLE: audio CD
- CD rate is 44,100 samples per second
  - 16-bit samples
  - Stereo uses 2 channels
- Number of bytes for 1 minute is
  - 2 X (16/8) X 60 X 44100 = 10.584 Mbytes

Discrete-Time Sinusoid

- Change x(t) into x[n] **DERIVATION**

$$x(t) = A\cos(\omega t + \varphi)$$

$$x[n] = x(nT_s) = A\cos(\omega n T_s + \varphi)$$

$$x[n] = A\cos((\omega T_s)n + \varphi)$$

$$x[n] = A\cos(\hat{\omega} n + \varphi)$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

DEFINE DIGITAL FREQUENCY

Digital Frequency $\hat{\omega}$

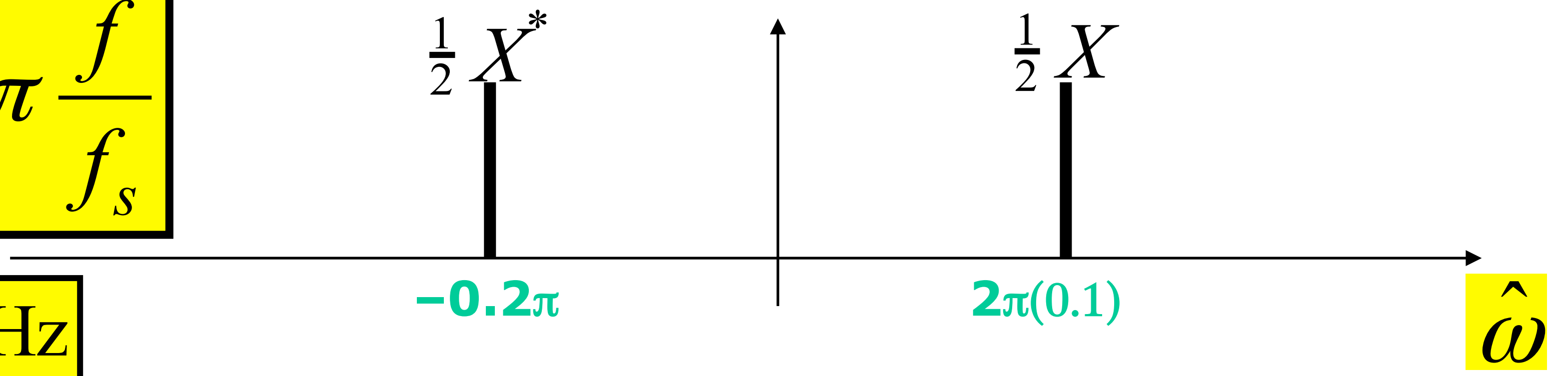
- $\hat{\omega}$ VARIES from **0** to **2π**, as f varies from 0 to the sampling frequency
- UNITS are radians, **not** rad/sec
  - DIGITAL FREQUENCY is NORMALIZED

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}$$

Spectrum (Digital)

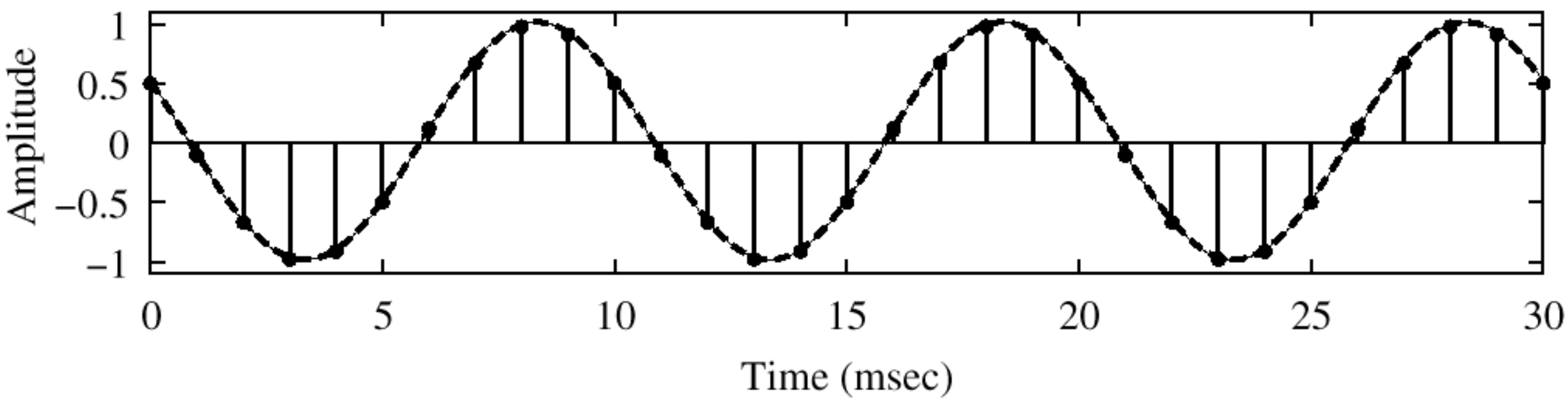
$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 1\,\text{kHz}$$



$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$

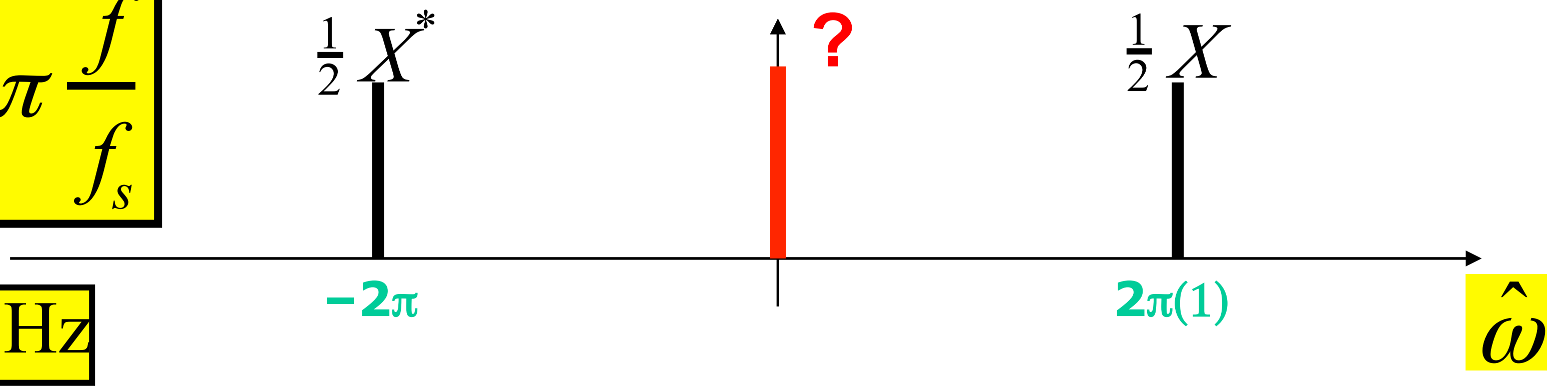
100-Hz Cosine Wave: Sampled with T_s = 1 msec (1000 Hz)



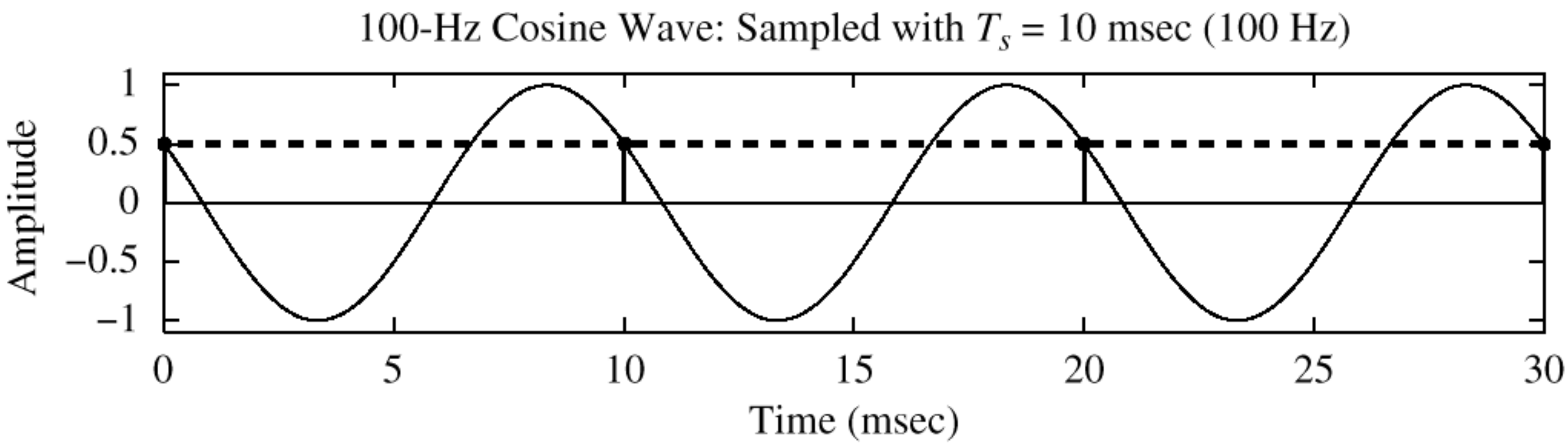
Spectrum (Digital) ???

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 100\text{ Hz}$$



$$x[n] = A\cos(2\pi(100)(n/100) + \varphi)$$



The Rest of the Story

- Spectrum of x[n] has more than one line for each complex exponential
  - Called **ALIASING**
  - **MANY SPECTRAL LINES**

- SPECTRUM is PERIODIC with period = **2π**
  - Because

$$A\cos(\hat{\omega}n+\varphi) = A\cos((\hat{\omega}+2\pi)n+\varphi)$$

Aliasing Derivation

- Other Frequencies give the same $\hat{\omega}$

$$x_1(t) = \cos(400\pi t) \quad \text{sampled at } f_s = 1000\,\text{Hz}$$

$$x_1[n] = \cos(400\pi \tfrac{n}{1000}) = \cos(0.4\pi n)$$

$$x_2(t) = \cos(2400\pi t) \quad \text{sampled at } f_s = 1000\,\text{Hz}$$

$$x_2[n] = \cos(2400\pi \tfrac{n}{1000}) = \cos(2.4\pi n)$$

$$x_2[n] = \cos(2.4\pi n) = \cos(0.4\pi n + 2\pi n) = \cos(0.4\pi n)$$

$$\Rightarrow x_2[n] = x_1[n]$$

$$2400\pi - 400\pi = 2\pi(1000)$$

Aliasing Derivation–2

- Other Frequencies give the same $\hat{\omega}$

$$\text{If } x(t) = A\cos(2\pi(f + \ell f_s)t + \varphi)$$

$$t \leftarrow \frac{n}{f_s}$$

$$\text{and we want: } x[n] = A\cos(\hat{\omega}n + \varphi)$$

$$\text{then: } \hat{\omega} = \frac{2\pi(f + \ell f_s)}{f_s} = \frac{2\pi f}{f_s} + \frac{2\pi \ell f_s}{f_s}$$

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi\ell$$

ALIASing SEASON FINALE

- ADDING f_s or $2f_s$ or $-f_s$ to the FREQ of x(t) gives exactly the same x[n]
  - The samples, $x[n] = x(n/f_s)$ are EXACTLY THE SAME VALUES

- GIVEN x[n], WE CAN'T DISTINGUISH f_o FROM $(f_o + f_s)$ or $(f_o + 2f_s)$

Normalized Frequency

- DIGITAL FREQUENCY

Normalized Radian Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi\ell$$

Normalized Cyclic Frequency

$$\hat{f} = \hat{\omega}/(2\pi) = f\,T_s = f/f_s$$

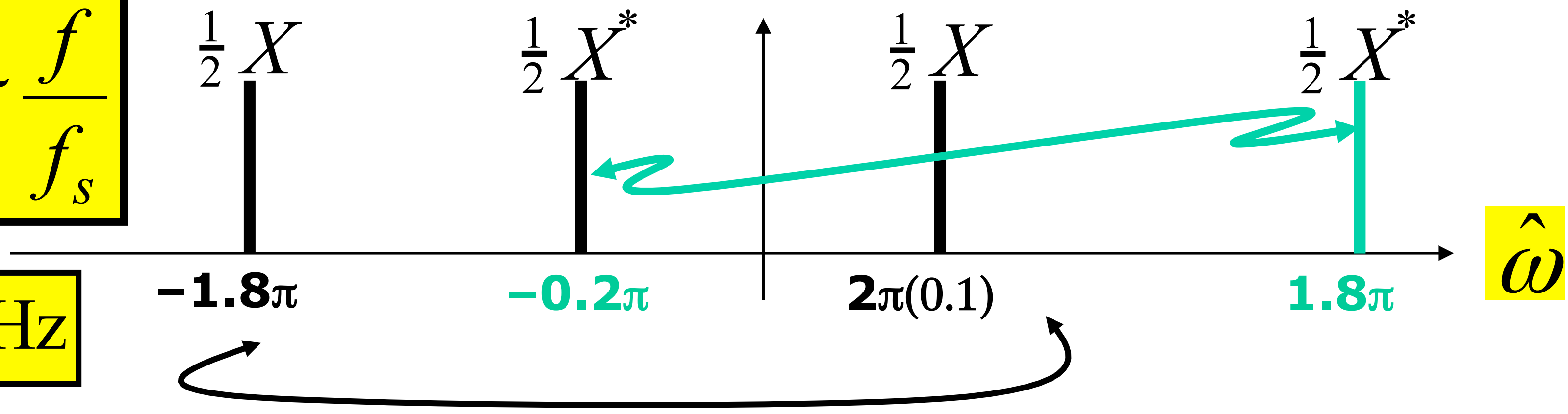
Spectrum for x[n]

- PLOT versus NORMALIZED FREQUENCY
- INCLUDE **ALL** SPECTRUM LINES
  - ALIASES
    - ADD MULTIPLES of 2π
    - SUBTRACT MULTIPLES of 2π
  - FOLDED ALIASES
    - (to be discussed later)
    - ALIASES of NEGATIVE FREQS

Spectrum (more lines)

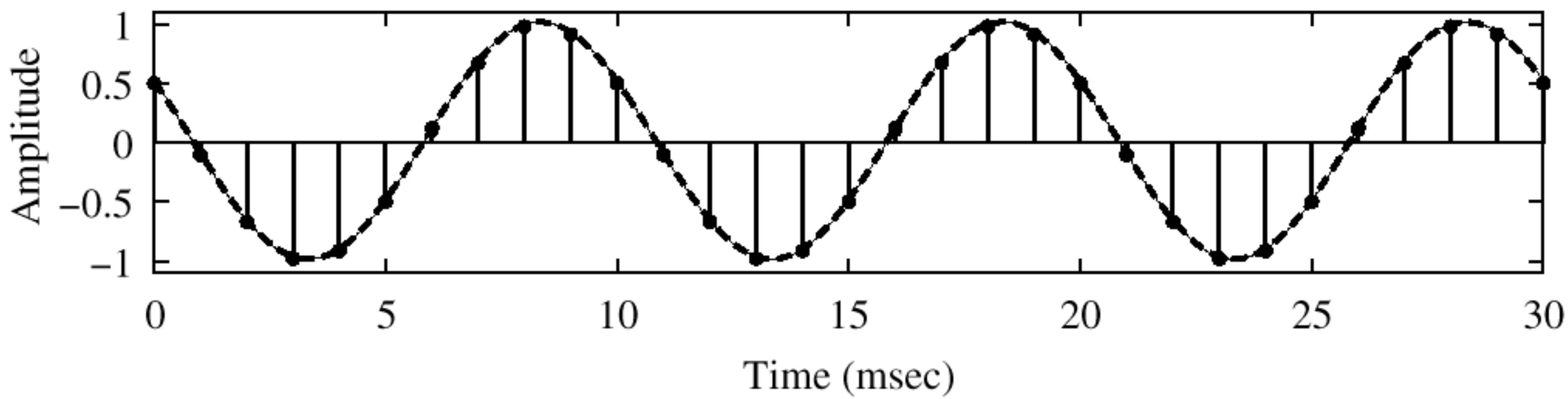
$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 1\,\text{kHz}$$



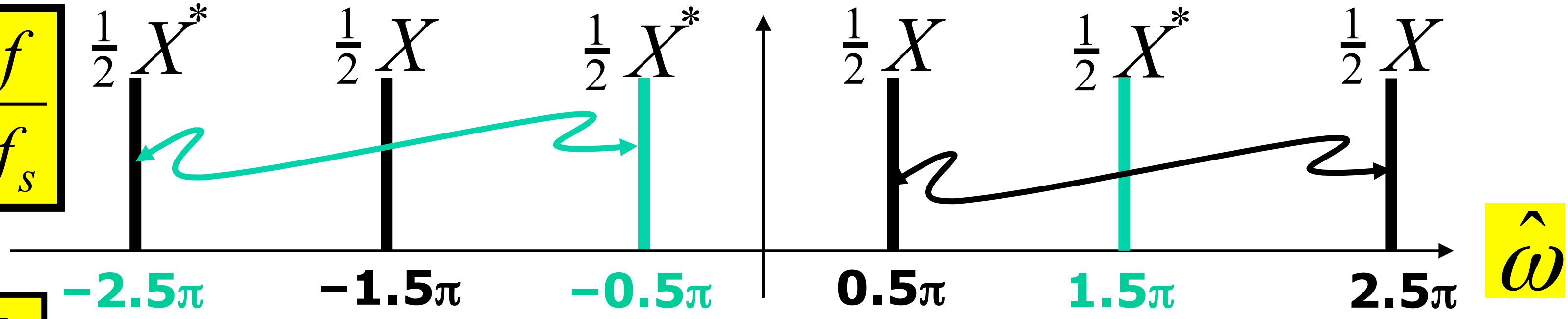
$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$

100-Hz Cosine Wave: Sampled with T_s = 1 msec (1000 Hz)



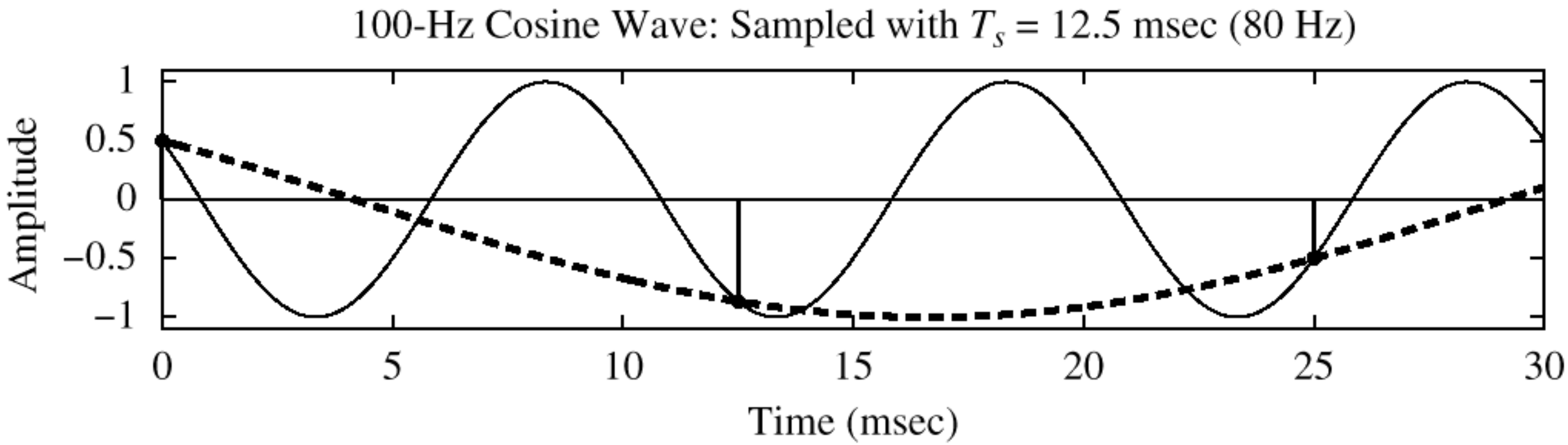
Spectrum (Aliasing Case)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

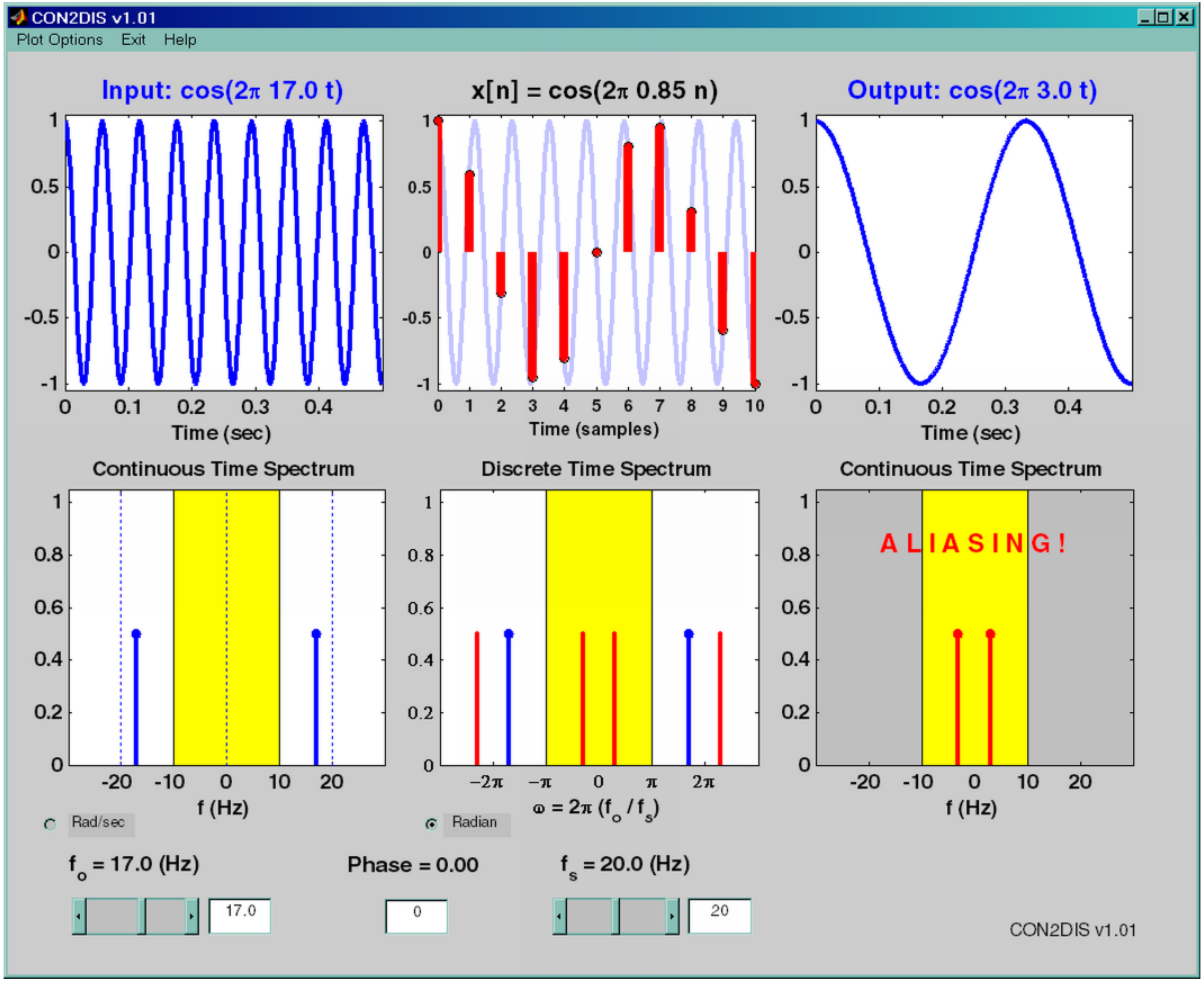


$$f_s = 80\,\text{kHz}$$

$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$



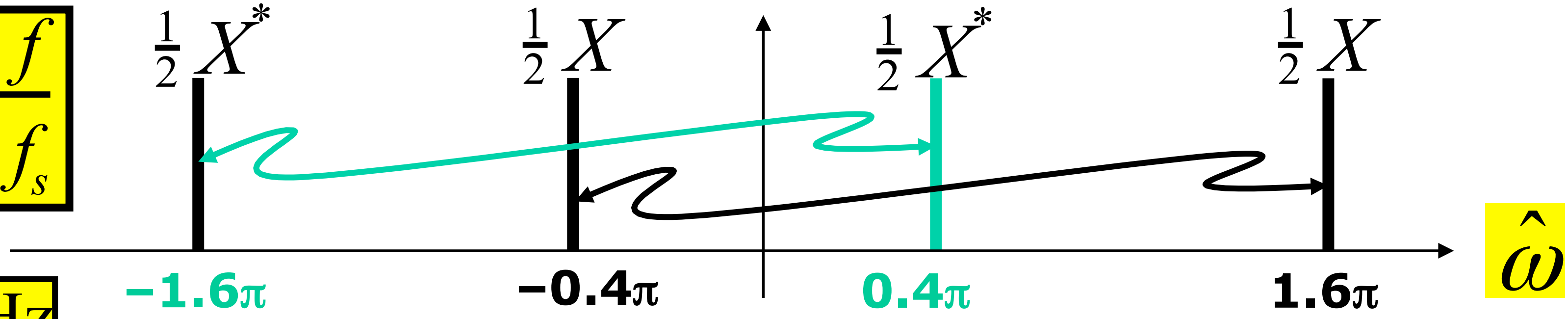
SAMPLING GUI (con2dis)



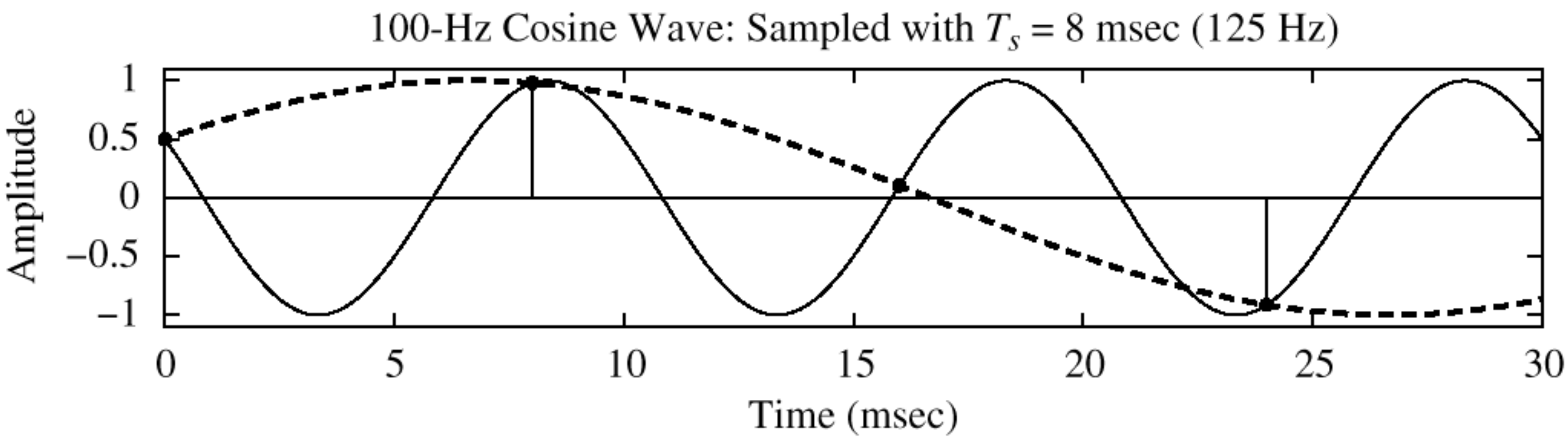
Spectrum (Folding Case)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 125\text{Hz}$$



$$x[n] = A\cos(2\pi(100)(n/125) + \varphi)$$



- Next week <> Section 4-4
  Section 4-5

  <> Lab 4