ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Circuits and Systems I

LECTURE #7

Bandlimited Reconstruction

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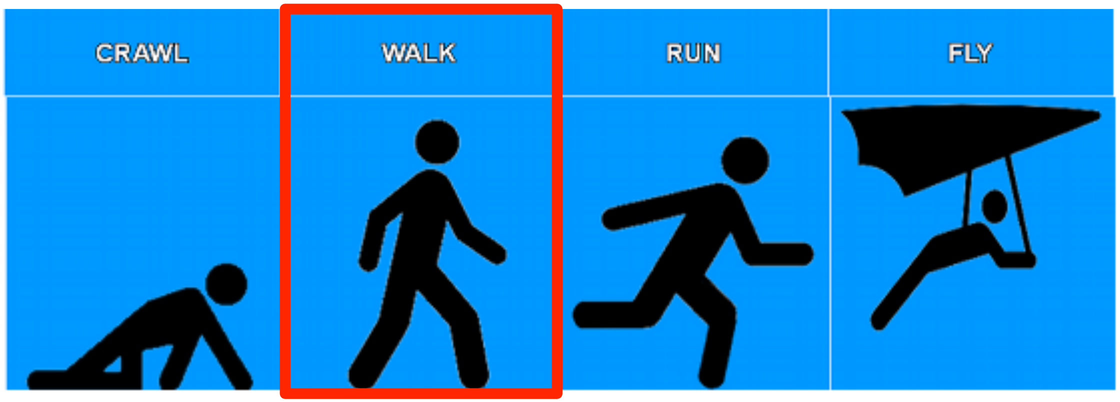
Outline - Today

- Today <> Section 4-4
  Section 4-5

- Next week <> BONUS EXAM REVIEW!

- Next lecture <> Section 5-1
  Section 5-2
  Section 5-3

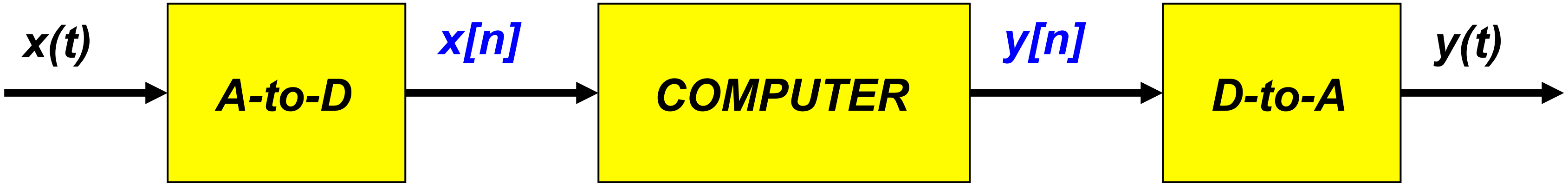
***CSI* Progress Level:**



Lecture Objectives

- FOLDING: a type of ALIASING
- DIGITAL-to-ANALOG CONVERSION is
  - Reconstruction from samples
    - SAMPLING THEOREM applies
  - Smooth **Interpolation**
- Mathematical Model of D-to-A
  - SUM of SHIFTED PULSES
    - Linear Interpolation example

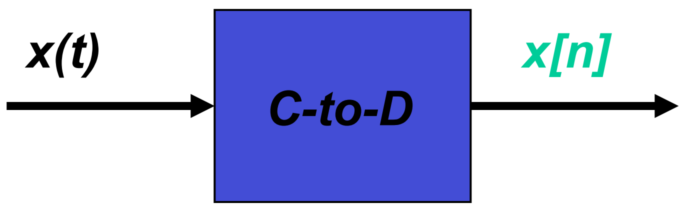
Signal Types



- A-to-D
  - Convert x(t) to numbers stored in memory
- D-to-A
  - Convert y[n] back to a "continuous-time" signal, y(t)
  - y[n] is called a "discrete-time" signal

Sampling x(t)

- UNIFORM SAMPLING at $t = nT_s$
  - IDEAL: $x[n] = x(nT_s)$



Shannon Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than $f_{\max}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\max}$.

Nyquist Rate

- "Nyquist Rate" Sampling
  - f_s > **TWICE** the HIGHEST Frequency in x(t)
  - "Sampling above the Nyquist rate"
- **BANDLIMITED SIGNALS**
  - DEF: x(t) has a HIGHEST FREQUENCY COMPONENT in its SPECTRUM
  - NON-BANDLIMITED EXAMPLE
    - TRIANGLE WAVE is NOT BANDLIMITED

SPECTRUM for x[n]

- INCLUDE ALL SPECTRUM LINES
  - ALIASES
    - ADD INTEGER MULTIPLES of 2π and -2π
  - FOLDED ALIASES
    - ALIASES of NEGATIVE FREQS
- PLOT versus NORMALIZED FREQUENCY
  - i.e., DIVIDE f_o by f_s

$$\hat{\omega} = 2\pi \frac{f}{f_s} + 2\pi\ell$$

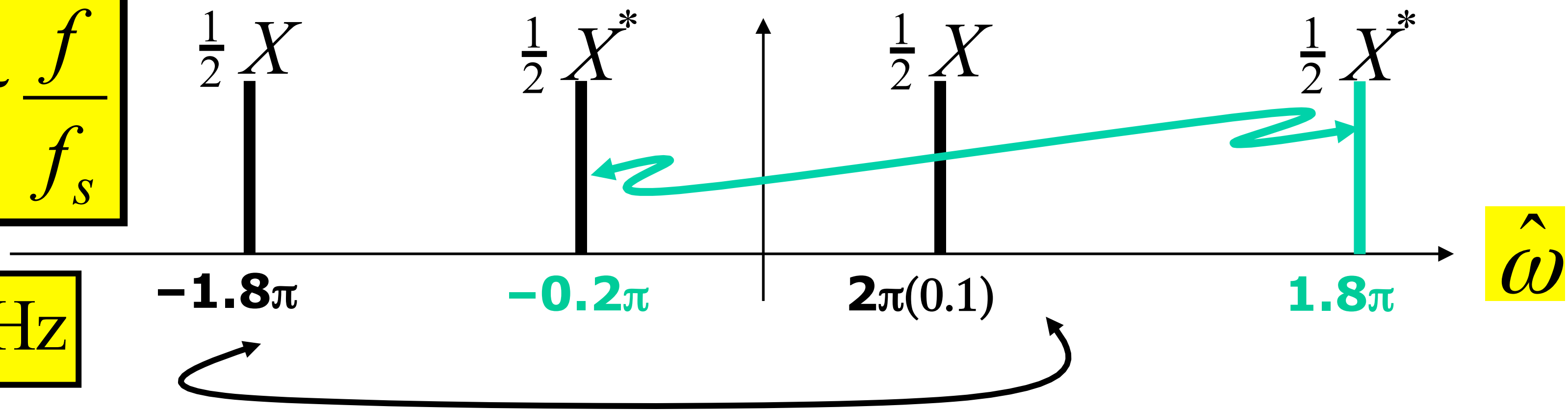
Example: Spectrum

- $x[n] = A\cos(0.2\pi n+\phi)$
- FREQS @ 0.2π and -0.2π
- ALIASES:
  - $\{2.2\pi, 4.2\pi, 6.2\pi, \ldots\}$ & $\{-1.8\pi, -3.8\pi, \ldots\}$
  - EX: $x[n] = A\cos(4.2\pi n+\phi)$
- ALIASES of NEGATIVE FREQ:
  - $\{1.8\pi, 3.8\pi, 5.8\pi, \ldots\}$ & $\{-2.2\pi, -4.2\pi \ldots\}$

Spectrum (More Lines)

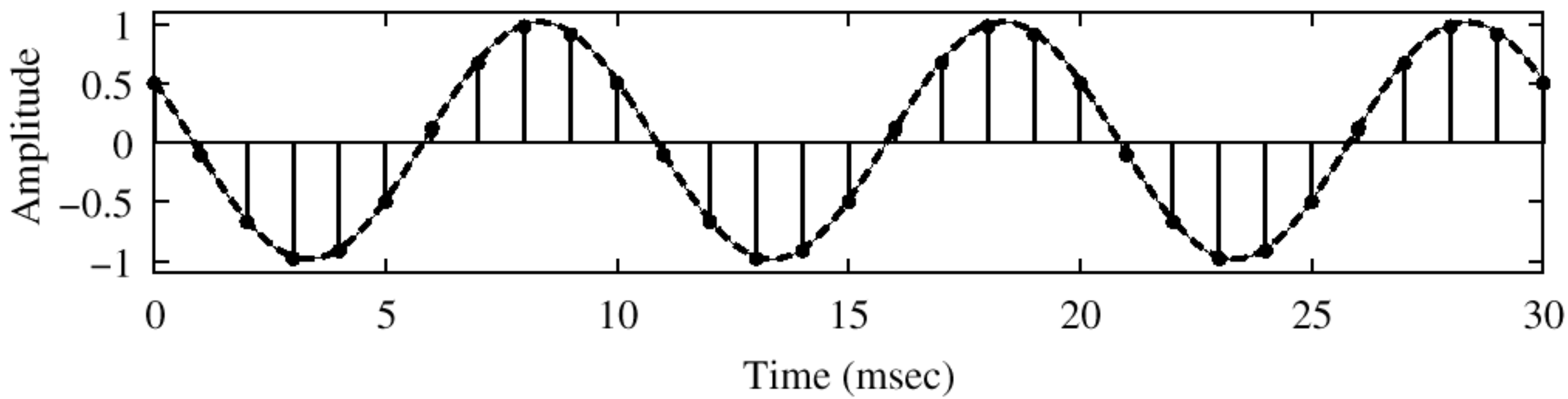
$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 1\,\text{kHz}$$



$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$

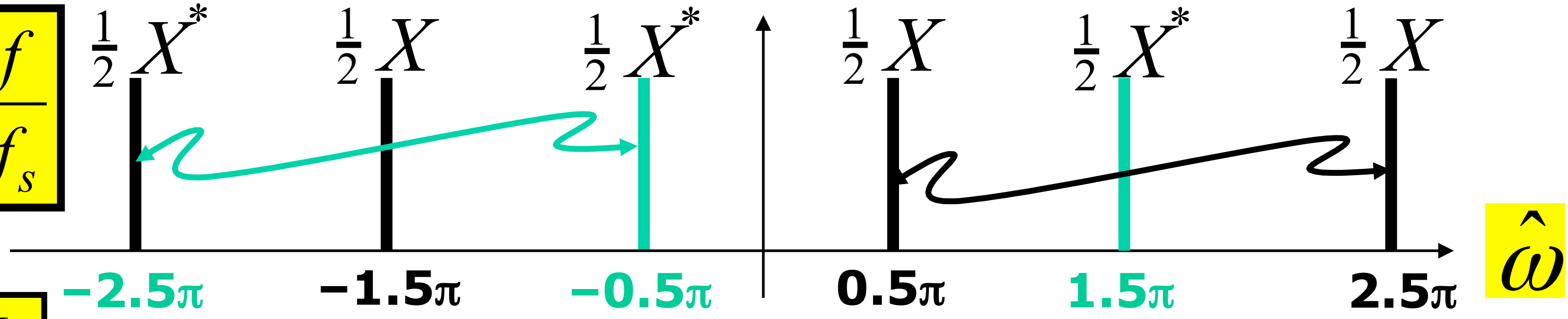
100-Hz Cosine Wave: Sampled with T_s = 1 msec (1000 Hz)



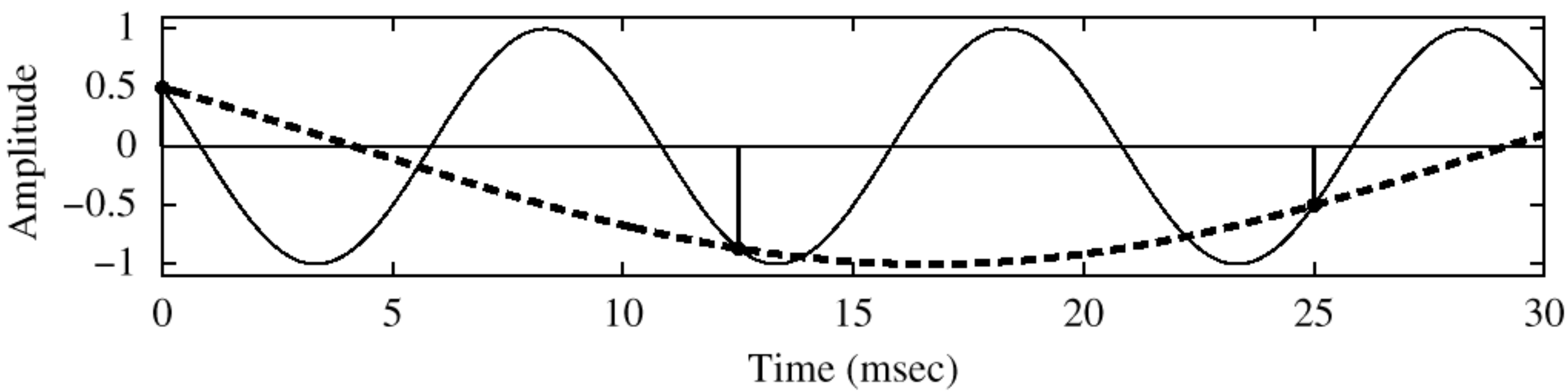
Spectrum (Aliasing Case)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 80\,\text{kHz}$$



$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$



Folding (a type of ALIASING)

- EXAMPLE: 3 different x(t); same x[n]

$$\hat{\omega} = 2\pi \frac{100}{1000} = 2\pi(0.1)$$

$$f_s = 1000$$

$$\cos(2\pi(100)t) \rightarrow \cos[2\pi(0.1)n]$$

$$\cos(2\pi(1100)t) \rightarrow \cos[2\pi(1.1)n] = \cos[2\pi(0.1)n]$$

$$\cos(2\pi(900)t) \rightarrow \cos[2\pi(0.9)n]$$

$$= \cos[2\pi(0.9)n - 2\pi n] = \cos[2\pi(-0.1)n] = \cos[2\pi(0.1)n]$$

- 900 Hz “folds” to 100 Hz when f_s=1kHz

Digital Frequency $\hat{\omega}$ Again

Normalized Radian Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi \ell$$

ALIASING

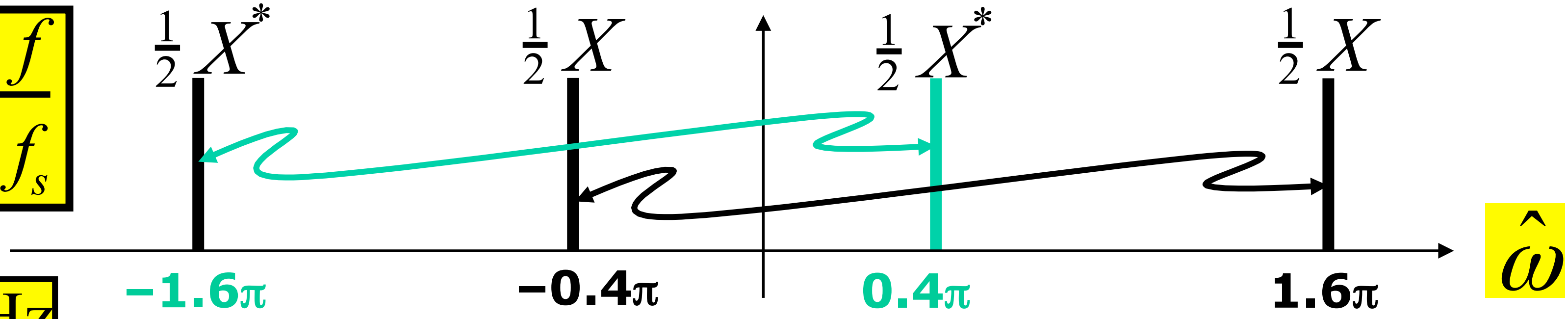
$$\hat{\omega} = \omega T_s = -\frac{2\pi f}{f_s} + 2\pi \ell$$

FOLDED ALIAS

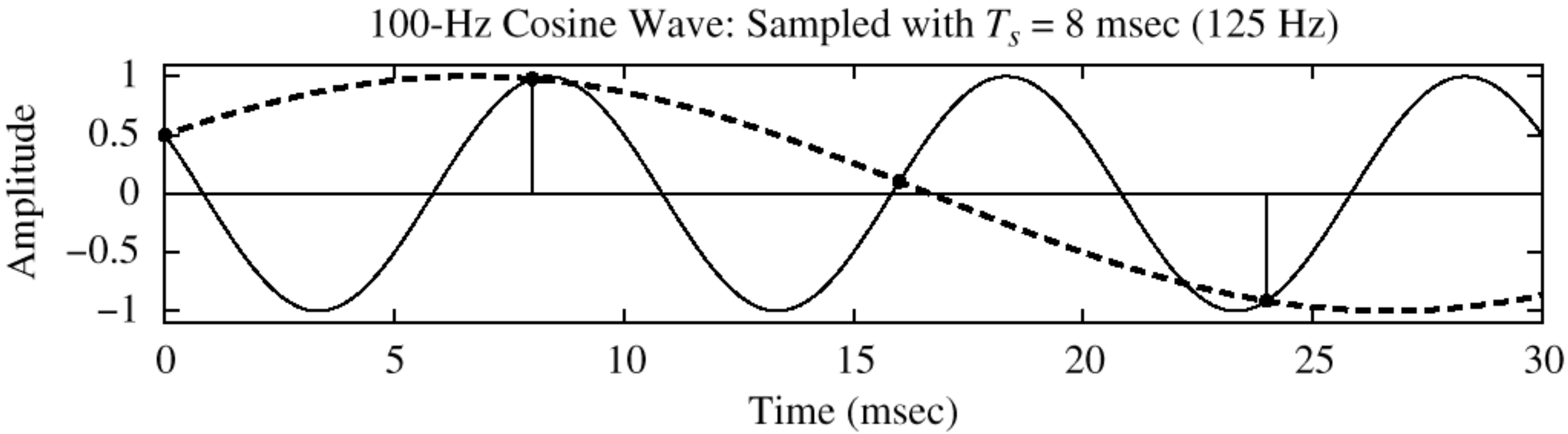
Spectrum (Folding Case)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

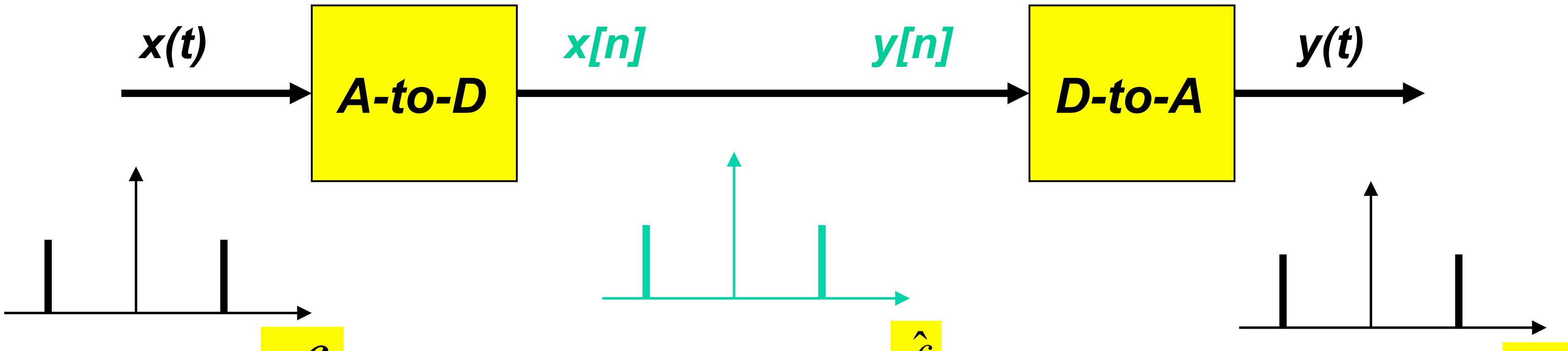
$$f_s = 125\text{Hz}$$



$$x[n] = A\cos(2\pi(100)(n/125) + \varphi)$$



Frequency Domains



f ω $\hat{f}$ $\hat{\omega}$ f

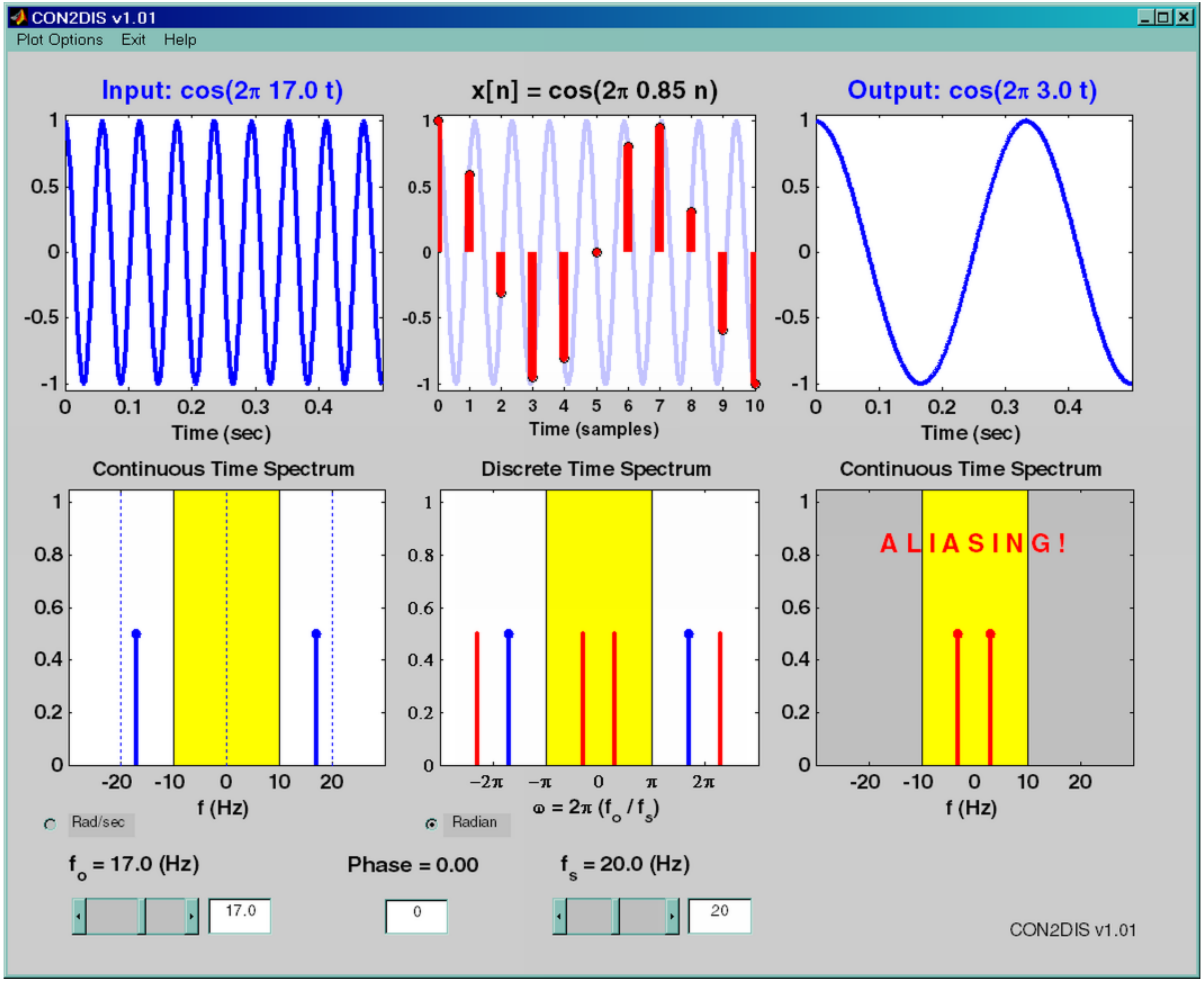
$$\hat{\omega} = 2\pi \frac{f}{f_s} + 2\pi\ell$$

$$f = \frac{\hat{\omega}}{2\pi} f_s$$

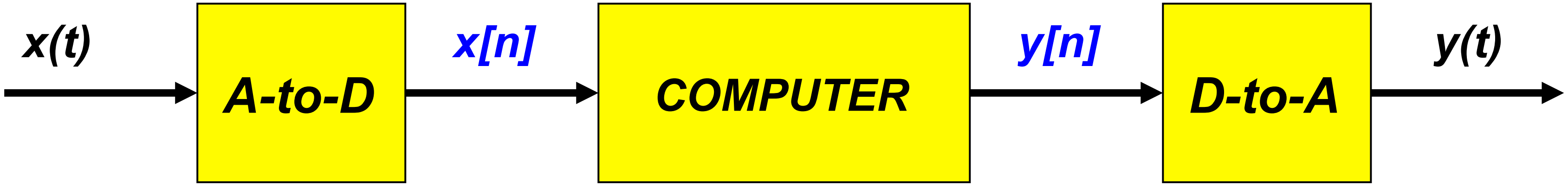
Demos from Chapter 4

- CD-ROM DEMOS
- SAMPLING DEMO (**con2dis GUI**)
  - Different Sampling Rates
    - Aliasing of a Sinusoid
- STROBE DEMO
  - Synthetic vs. Real
  - Television SAMPLES at 30 fps in the US / 25 fps in EU
- Sampling & Reconstruction

SAMPLING GUI (con2dis)



D-to-A Reconstruction



- Create continuous y(t) from y[n]
  - **IDEAL**
    - If you have formula for y[n]
  - Replace n in y[n] with $f_s t$
  - $y[n] = A\cos(0.2\pi n+\phi)$ with $f_s = 8000$ Hz
  - $y(t) = A\cos(2\pi(800)t+\phi)$

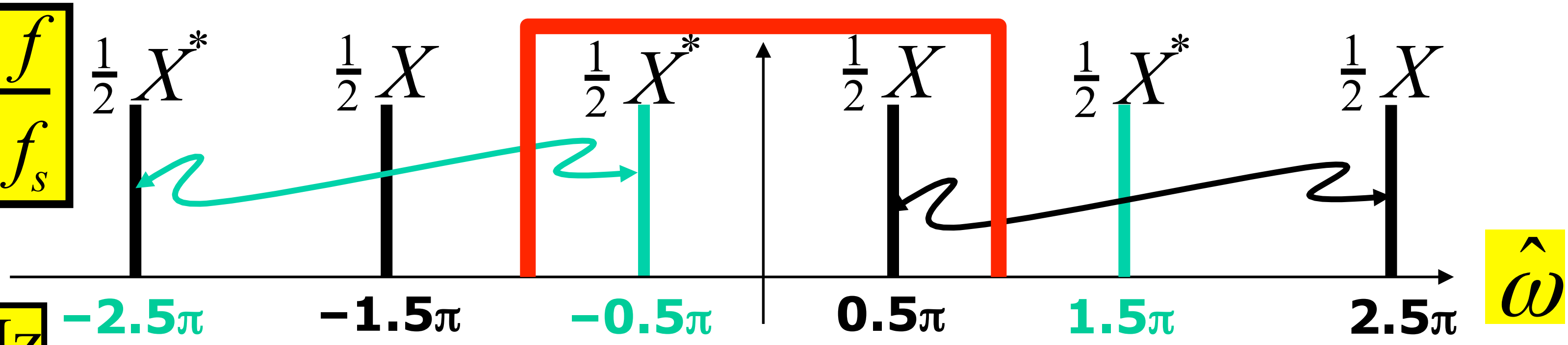
D-to-A is AMBIGUOUS !

- ALIASING
  - Given y[n], which y(t) do we pick ? ? ?
  - INFINITE NUMBER of y(t)
    - PASSING THRU THE SAMPLES, y[n]
  - D-to-A RECONSTRUCTION MUST CHOOSE ONE OUTPUT
- RECONSTRUCT THE SMOOTHEST ONE
  - THE LOWEST FREQ, if y[n] = sinusoid

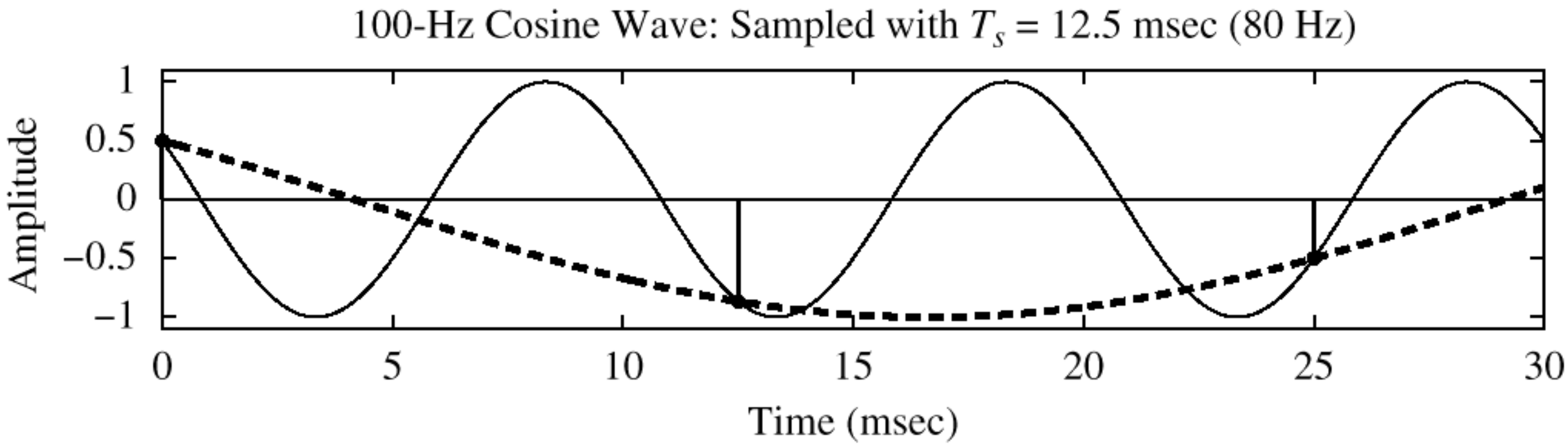
Spectrum (Aliasing Case)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 80\,\mathrm{Hz}$$

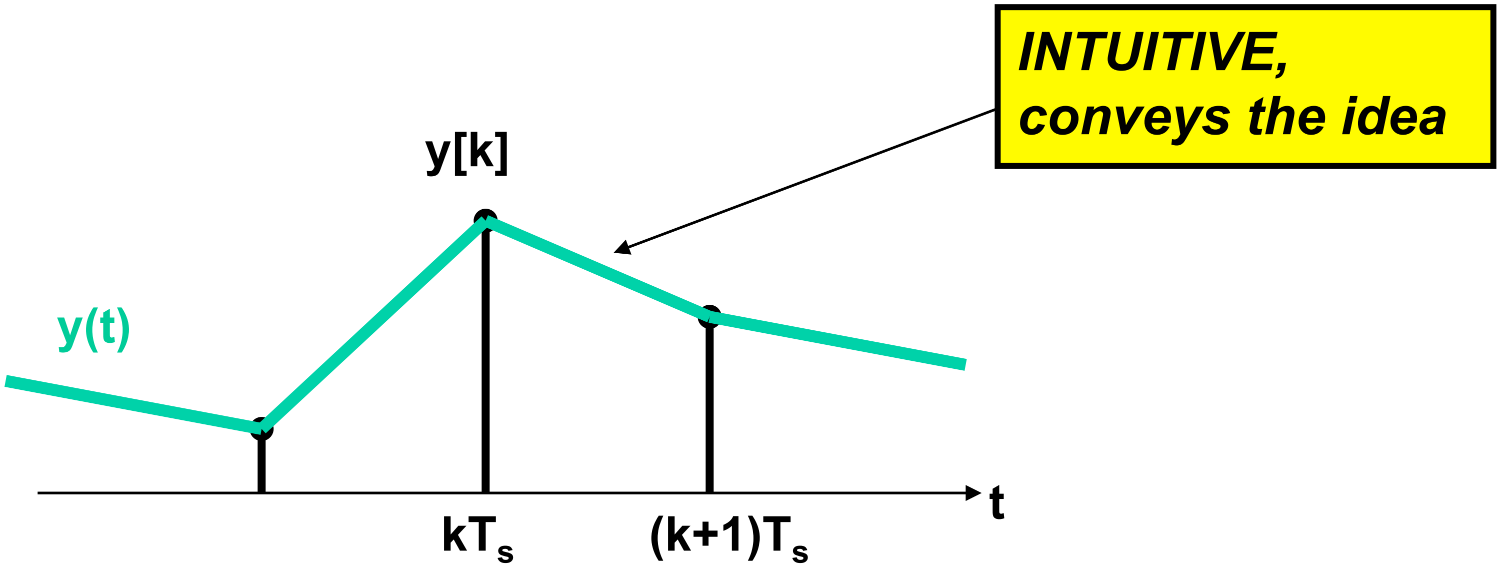


$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$



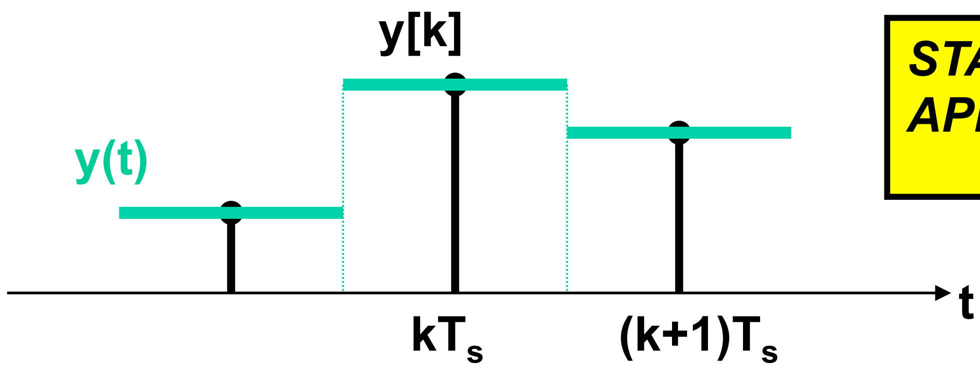
Reconstruction (D-to-A)

- CONVERT STREAM of NUMBERS to x(t)
- "CONNECT THE DOTS"
- INTERPOLATION



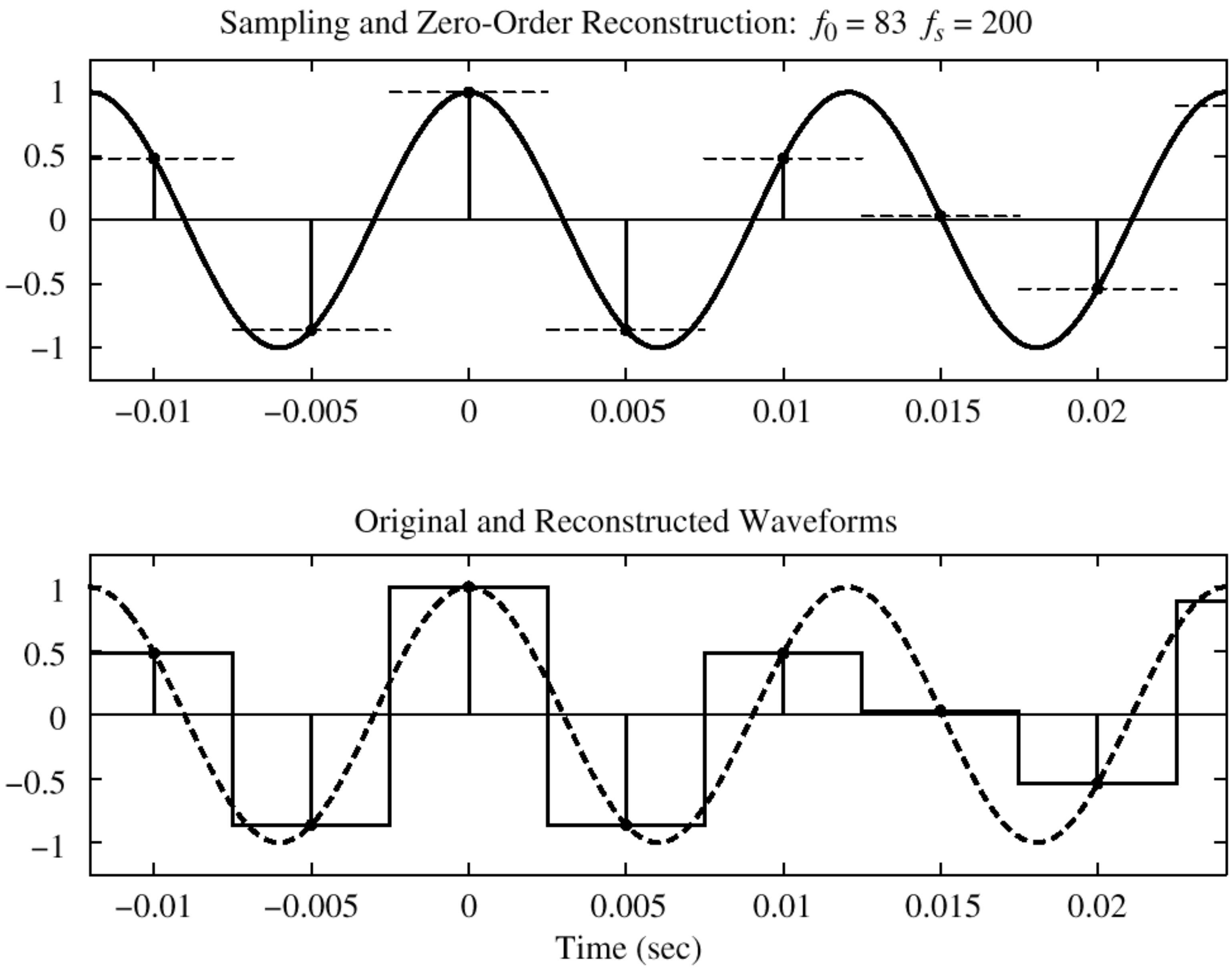
Sample and Hold Device

- CONVERT y[n] to y(t)
  - y[k] should be the value of y(t) at $t = kT_s$
  - Make y(t) equal to y[k] for
    - $kT_s - 0.5T_s < t < kT_s + 0.5T_s$

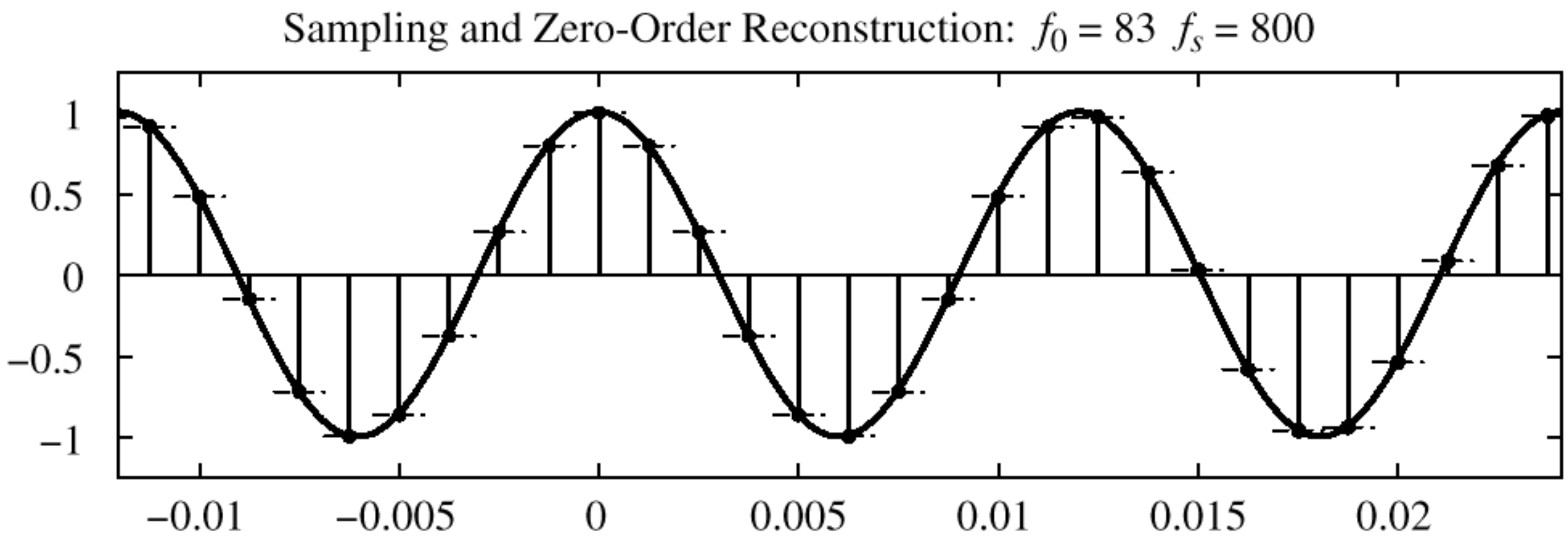


STAIR-STEP APPROXIMATION

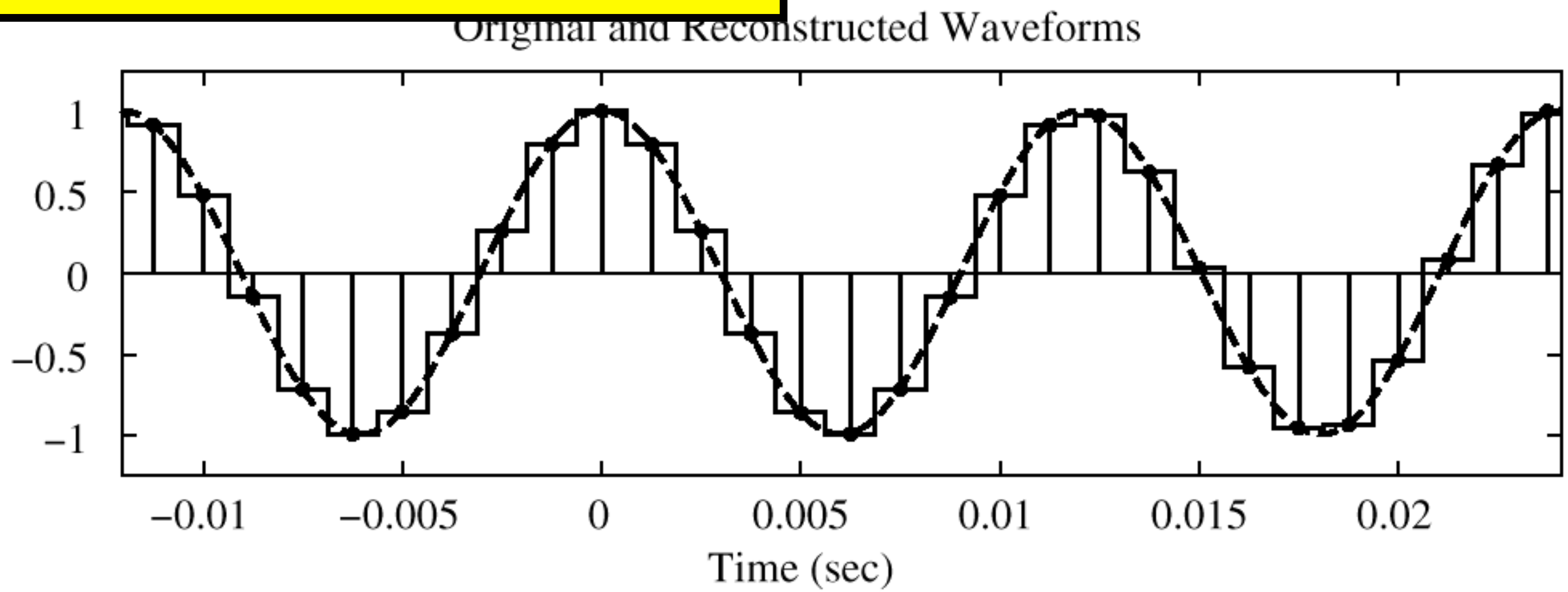
Square Pulse Case



Over-Sampling Case



EASIER TO RECONSTRUCT



Mathematical Model for D-to-A

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$

SQUARE PULSE:

$$p(t) = \begin{cases} 1 & -\frac{1}{2}T_s < t \leq \frac{1}{2}T_s \\ 0 & \text{otherwise} \end{cases}$$

Expand the Summation

$$\sum_{n=-\infty}^{\infty} y[n]p(t - nT_s) =$$

$$\ldots + y[0]p(t) + y[1]p(t - T_s) + y[2]p(t - 2T_s) + \ldots$$

- SUM of SHIFTED PULSES $p(t\text{-}nT_s)$
  - "WEIGHTED" by y[n]
  - CENTERED at $t=nT_s$
  - SPACED by T_s
    - RESTORES "REAL TIME"

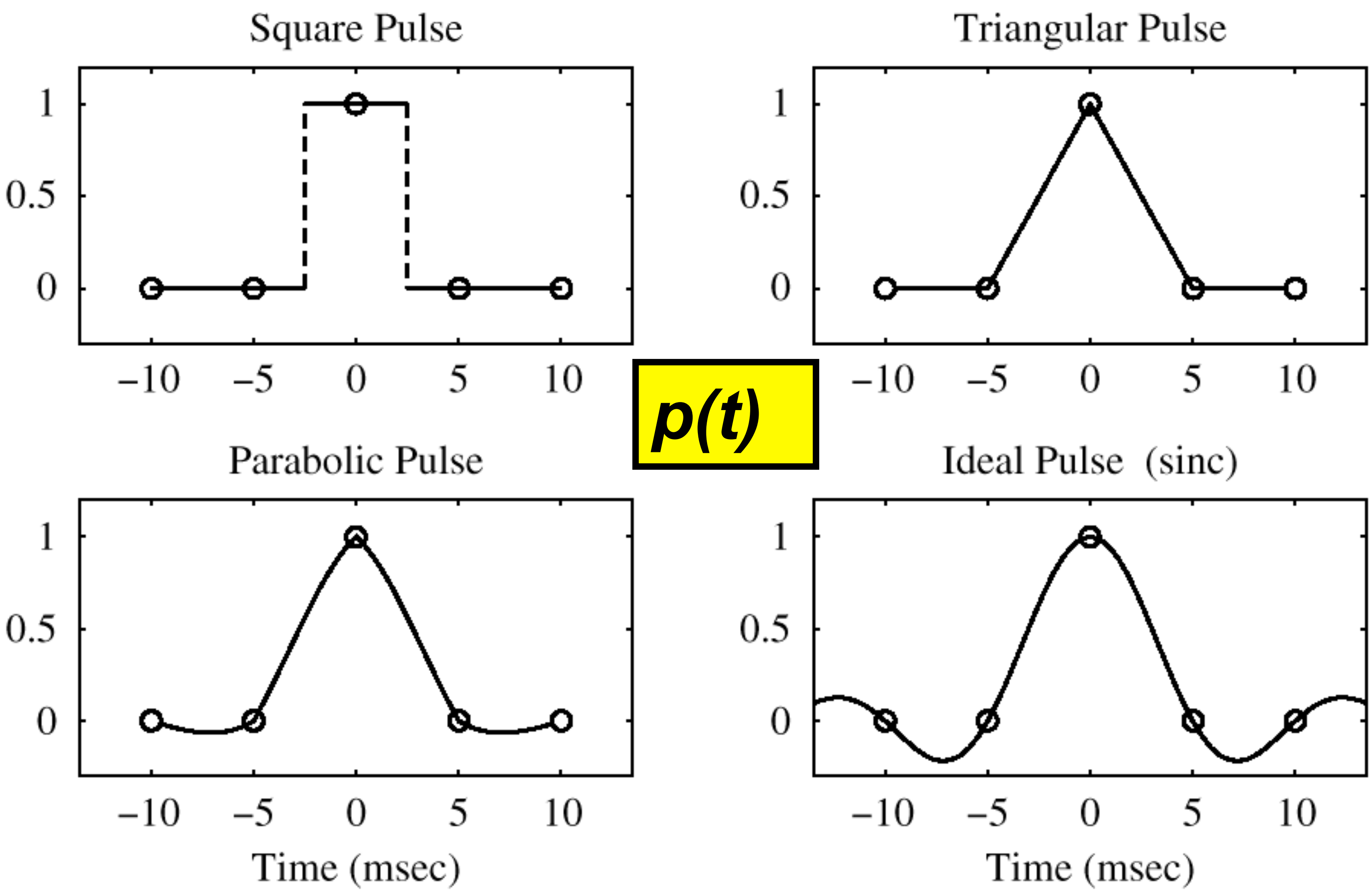
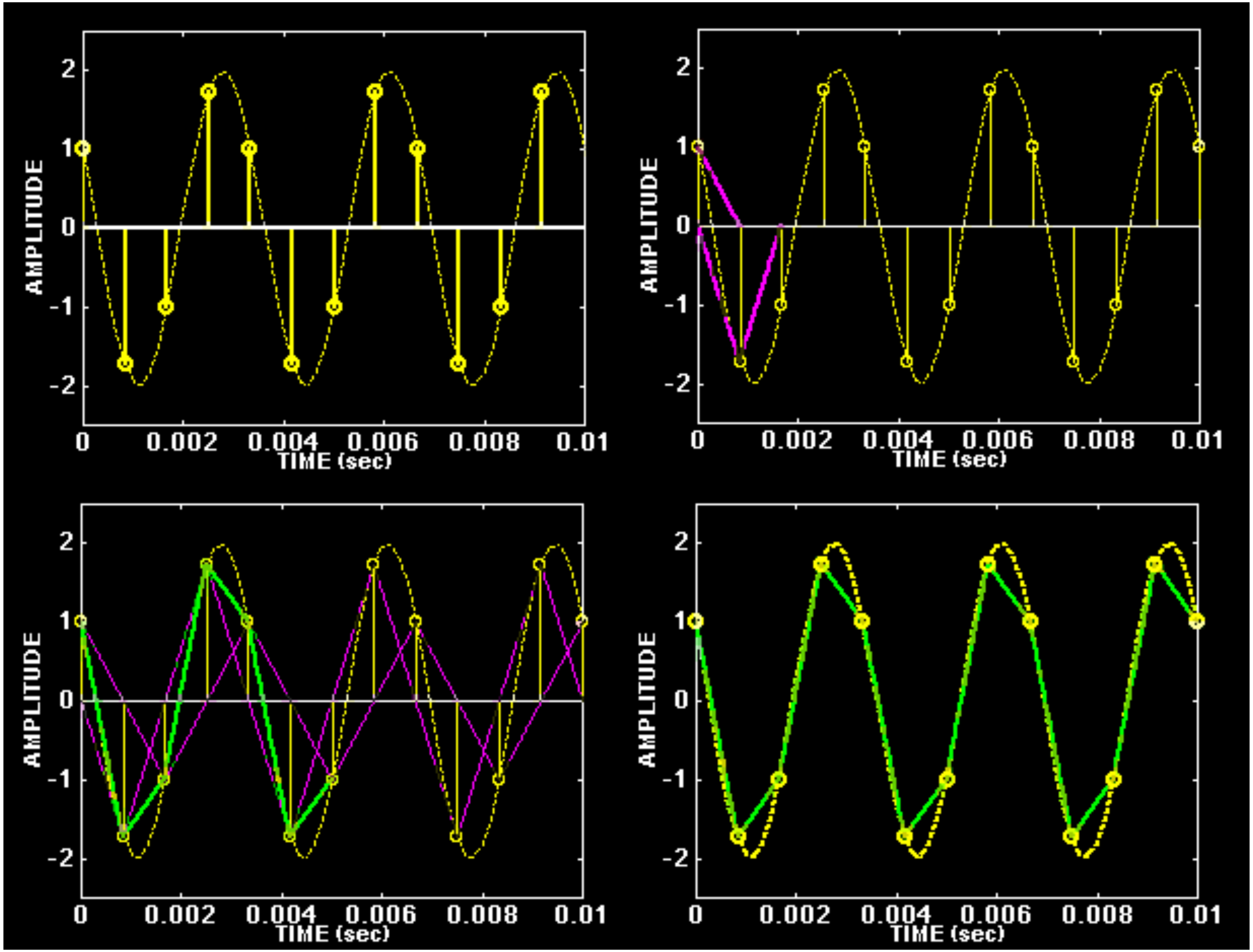


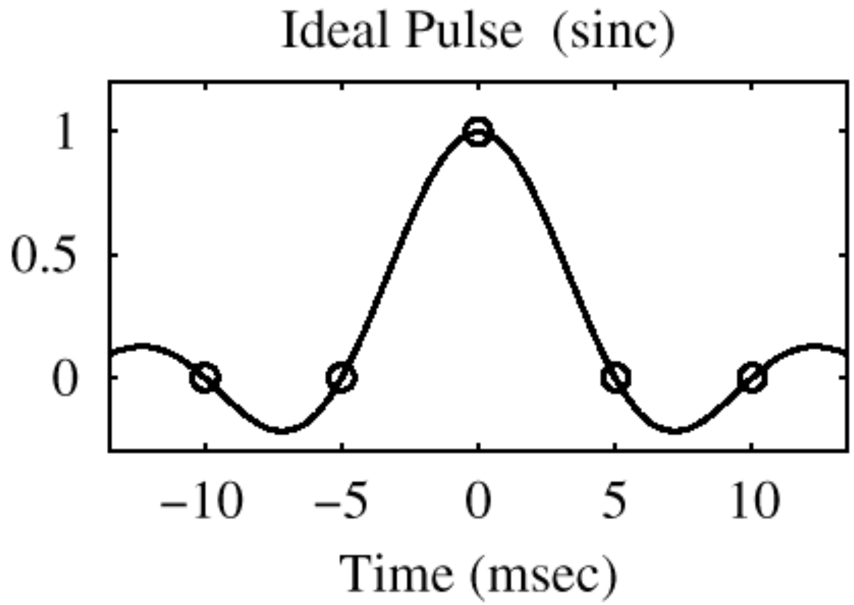
Figure 4.17 Four different pulses for D-to-C conversion. The sampling period is $T_s = 0.005$, i.e., $f_s = 200$ Hz. Note that the duration of each pulse is approximately one or two times T_s.

TRIANGULAR PULSE (2X)



Optimal Pulse?

CALLED "BANDLIMITED INTERPOLATION"



$$p(t) = \frac{\sin\frac{\pi t}{T_s}}{\frac{\pi t}{T_s}} \quad \text{for} -\infty < t < \infty$$

$$p(t) = 0 \quad \text{for } t = \pm T_s, \pm 2T_s, \ldots$$

- Next week <> Bonus Exam Review

- **LAB TIME NOW**