Mathematics of Data: From Theory to Computation

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Lecture 3: Convex analysis and complexity

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Outline

- This lecture
  1. Learning as an optimization problem
  2. Basic concepts in convex analysis
  3. Complexity theory review
     - Asymptotic notation
     - Computational complexity
     - Hardness result: certifying optimality in non-convex problems
- Next lecture
  1. Unconstrained convex optimization: the basics
  2. Gradient descent methods

Recommended reading

- Chapter 2 & 3 in S. Boyd, and L. Vandenberghe, *Convex Optimization*, Cambridge Univ. Press, 2009.
- Appendices A & B in D. Bertsekas, *Nonlinear Programming*, Athena Scientific, 1999.
- Chapter 3 & 34 in Cormen, Thomas H., et al. Introduction to algorithms. Vol. 2. Cambridge: MIT press, 2001.
- Sections 3.1, 3.2, 5.3, 6.3, 7.2-7.5 in Sipser, Michael. Introduction to the Theory of Computation. Cengage Learning, 2012.

Motivation

Motivation

- The first part of this recitation introduces basic notions in convex analysis.
- The second part is intended to help you understand some concepts in the theory of computation that you will encounter in discussions concerned with efficient computation, and some of the notation involved.

Practical Issues

Recall from Lecture 2:
Given an estimator $\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathcal{X}} \{F(\mathbf{x})\}$ of $\mathbf{x}^\natural$, we discussed two key questions:

1. Is the formulation reasonable?
2. What is the role of the data size?

Consider the estimation error in the ℓ_2-norm: $\left\|\hat{\mathbf{x}} - \mathbf{x}^\natural\right\|_2$.

- Is $\left\|\hat{\mathbf{x}} - \mathbf{x}^\natural\right\|_2$ enough to evaluate the performance of the estimator $\hat{\mathbf{x}}$?

Practical Issues

No, because in general we can only *numerically approximate* the solution of

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x} \in \mathbb{R}^p} \{F(\mathbf{x})\}.$$

Implementation

How do we *numerically approximate* $\hat{\mathbf{x}}$?

Practical performance

Denote the numerical approximation by $\mathbf{x}^\star_\epsilon$. The practical performance is governed by

$$\left\|\mathbf{x}^\star_\epsilon - \mathbf{x}^\natural\right\|_2 \leq \underbrace{\|\mathbf{x}^\star_\epsilon - \hat{\mathbf{x}}\|_2}_{\text{approximation error}} + \underbrace{\left\|\hat{\mathbf{x}} - \mathbf{x}^\natural\right\|_2}_{\text{statistical error}}.$$

How do we evaluate $\|\mathbf{x}^\star_\epsilon - \hat{\mathbf{x}}\|_2$?

- The ϵ-approximation solution $\mathbf{x}^\star_\epsilon$ will be defined rigorously in the later lectures.

Practical issues

How do we *numerically approximate* $\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^p} \{F(\mathbf{x})\}$ for a given F?

General idea of an optimization algorithm

Guess a solution, and then *refine* it based on *oracle information*.

Repeat the procedure until the result is *good enough*.

How do we evaluate the approximation error $\|\mathbf{x}^\star_\epsilon - \hat{\mathbf{x}}\|_2$?

General concept about the approximation error

It depends on the *characteristics* of the function F and the chosen numerical *optimization algorithm*.

Need for convex analysis

General idea of an optimization algorithm

Guess a solution, and then *refine* it based on *oracle information*.

Repeat the procedure until the result is *good enough*.

General concept about the approximation error

It depends on the *characteristics* of the function F and the chosen numerical *optimization algorithm*.

Role of convexity

Convexity provides a key optimization framework in obtaining numerical approximations at theoretically well-understood computational costs.

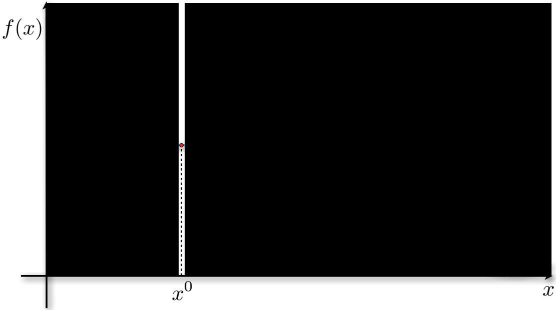
To precisely understand these ideas, we need to understand basics of *convex analysis*.

Challenges for an iterative optimization algorithm

Problem

Find the minimum $x^\star$ of $f(x)$, given starting point x^0 based on only local information.

- **Fog of war**

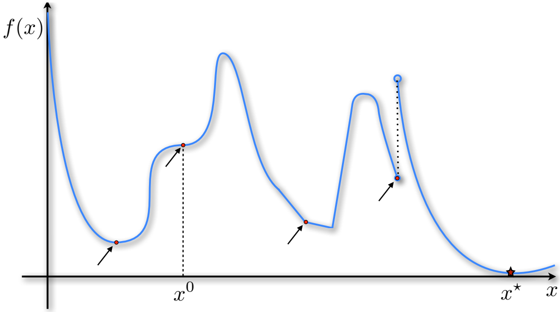


Challenges for an iterative optimization algorithm

Problem

Find the minimum $x^\star$ of $f(x)$, given starting point x^0 based on only local information.

- Fog of war, non-differentiability, discontinuities, local minima, stationary points...



Challenges for an iterative optimization algorithm

Problem

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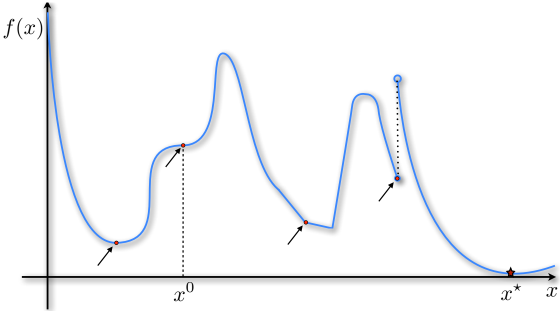


We need a key structure on the function local minima: Convexity.

Basics of functions

Definition (Function)

A function f with domain $\mathcal{Q} \subseteq \mathbb{R}^p$ and codomain $\mathcal{U} \subseteq \mathbb{R}$ is denoted as:

$$f \ : \ \mathcal{Q} \to \mathcal{U}.$$

The domain $\mathcal{Q}$ represents the set of values in $\mathbb{R}^p$ on which f is defined and is denoted as $\mathrm{dom}(f) \equiv \mathcal{Q} = \{\mathbf{x} : -\infty < f(\mathbf{x}) < +\infty\}$. The codomain $\mathcal{U}$ is the set of function values of f for any input in $\mathcal{Q}$.

Continuity in functions

Definition (Continuity)

Let $f : \mathcal{Q} \to \mathbb{R}$ where $\mathcal{Q} \subseteq \mathbb{R}^p$. Then, f is a continuous function over its domain $\mathcal{Q}$ if and only if

$$\lim_{\mathbf{x} \to \mathbf{y}} f(\mathbf{x}) = f(\mathbf{y}), \quad \forall \mathbf{y} \in \mathcal{Q},$$

i.e., the limit of f—as $\mathbf{x}$ approaches $\mathbf{y}$—exists and is equal to $f(\mathbf{y})$.

Definition (Class of continuous functions)

We denote the class of continuous functions f over the domain $\mathcal{Q}$ as $f \in \mathcal{C}(\mathcal{Q})$.

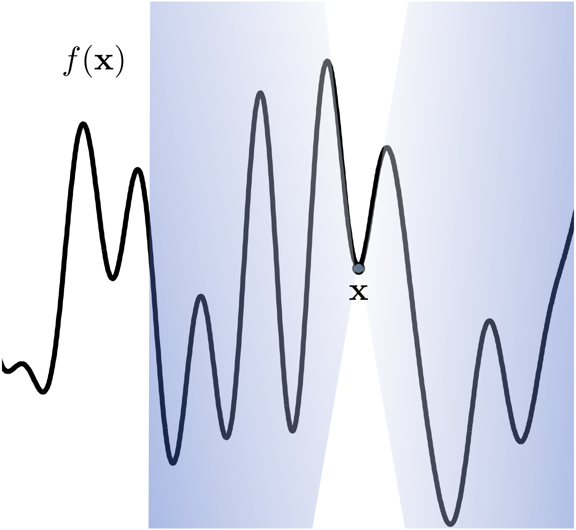
Definition (Lipschitz continuity)

Let $f : \mathcal{Q} \to \mathbb{R}$ where $\mathcal{Q} \subseteq \mathbb{R}^p$. Then, f is called Lipschitz continuous if there exists a constant value $K \geq 0$ such that:

$$|f(\mathbf{y}) - f(\mathbf{x})| \leq K \|\mathbf{y} - \mathbf{x}\|_2, \quad \forall \mathbf{x},\ \mathbf{y} \in \mathcal{Q}.$$

- "Small" changes in the input result into "small" changes in the function values.

Continuity in functions



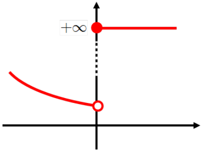
Lower semi-continuity

Definition

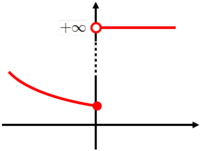
A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous (l.s.c.) if

$$\liminf_{\mathbf{x} \to \mathbf{y}} f(\mathbf{x}) \geq f(\mathbf{y}), \quad \text{for any } \mathbf{y} \in \text{dom}(f).$$

$$f(x) = \begin{cases} e^{-x}, & \text{if } x < 0 \\ +\infty, & \text{if } x \geq 0 \end{cases}$$



$$f(x) = \begin{cases} e^{-x}, & \text{if } x \leq 0 \\ +\infty, & \text{if } x > 0 \end{cases}$$



Unless stated otherwise, we only consider l.s.c. functions.

Lower semi-continuity

Definition

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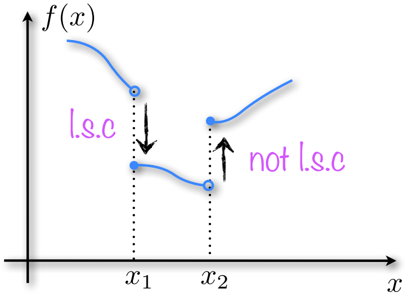
$$\liminf_{\mathbf{x}\to\mathbf{y}} f(\mathbf{x}) \geq f(\mathbf{y}), \quad \text{for any } \mathbf{y} \in \text{dom}(f).$$

- **Intuition:** A lower semi-continuous function *only jumps down.*



Differentiability in functions

- We use $\nabla f(\mathbf{x})$ to denote the *gradient* of f at $\mathbf{x} \in \mathbb{R}^p$ such that:

$$\nabla f(\mathbf{x}) = \sum_{i=1}^{p} \frac{\partial f}{\partial x_i}\mathbf{e}_i = \left[\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_p}\right]^T$$

Example: $f(\mathbf{x}) = \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2^2$

$\nabla f(\mathbf{x}) = -2\mathbf{A}^T(\mathbf{b} - \mathbf{A}\mathbf{x})$.

Definition (Differentiability)

Let $f \in \mathcal{C}(\mathcal{Q})$ where $\mathcal{Q} \subseteq \mathbb{R}^p$. Then, f is a k-times continuously differentiable on $\mathcal{Q}$ if its partial derivatives up to k-th order exist and are continuous $\forall \mathbf{x} \in \mathcal{Q}$.

Definition (Class of differentiable functions)

We denote the class of k-times continuously differentiable functions f on $\mathcal{Q}$ as $f \in \mathcal{C}^k(\mathcal{Q})$.

- In the special case of $k = 2$, we dub $\nabla^2 f(\mathbf{x})$ the **Hessian** of $f(\mathbf{x})$, where $[\nabla^2 f(\mathbf{x})]_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.
- We have $\mathcal{C}^q(\mathcal{Q}) \subseteq \mathcal{C}^k(\mathcal{Q})$ where $q \leq k$. For example, a twice differentiable function is also once differentiable.
- For the case of complex-valued matrices, we refer to the Matrix Cookbook online.

Differentiability in functions

- Some examples:

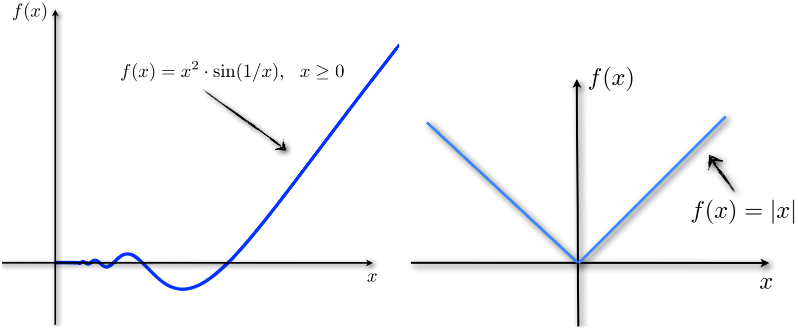


Figure: **(Left panel)** ∞-times continuously differentiable function in $\mathbb{R}$. **(Right panel)** Non-differentiable $f(x) = |x|$ in $\mathbb{R}$.

Stationary points of differentiable functions

Definition (Stationary point)

A point $\bar{\mathbf{x}}$ is called a stationary point of a twice differentiable function $f(\mathbf{x})$ if

$$\nabla f(\bar{\mathbf{x}}) = \mathbf{0}.$$

Definition (Local minima, maxima, and saddle points)

Let $\bar{\mathbf{x}}$ be a stationary point of a twice differentiable function $f(\mathbf{x})$.

- If $\nabla^2 f(\bar{\mathbf{x}}) \succ 0$, then the point $\bar{\mathbf{x}}$ is called a local minimum.
- If $\nabla^2 f(\bar{\mathbf{x}}) \prec 0$, then the point $\bar{\mathbf{x}}$ is called a local maximum.
- If $\nabla^2 f(\bar{\mathbf{x}}) = 0$, then the point $\bar{\mathbf{x}}$ can be a saddle point depending on the sign change.

Stationary points of smooth functions contd.

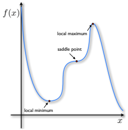
Intuition

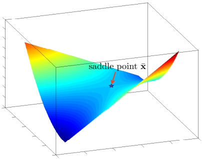
Recall Taylor's theorem for the function f around $\bar{\mathbf{x}}$ for all $\mathbf{y}$ that satisfy $\|\mathbf{y}-\bar{\mathbf{x}}\|_2 \leq r$ in a local region with radius r as follows

$$f(\mathbf{y}) = f(\bar{\mathbf{x}}) + \langle \nabla f(\bar{\mathbf{x}}), \mathbf{y}-\bar{\mathbf{x}}\rangle + \frac{1}{2}(\mathbf{y}-\bar{\mathbf{x}})^T \nabla^2 f(\mathbf{z})(\mathbf{y}-\bar{\mathbf{x}}),$$

where $\mathbf{z}$ is a point between $\bar{\mathbf{x}}$ and $\mathbf{y}$. When $r \to 0$, the second-order term becomes $\nabla^2 f(\mathbf{z}) \to \nabla^2 f(\bar{\mathbf{x}})$. Since $\nabla f(\bar{\mathbf{x}}) = 0$, Taylor's theorem leads to

- $f(\mathbf{y}) > f(\bar{\mathbf{x}})$ when $\nabla^2 f(\bar{\mathbf{x}}) \succ 0$. Hence, the point $\bar{\mathbf{x}}$ is a local minimum.
- $f(\mathbf{y}) < f(\bar{\mathbf{x}})$ when $\nabla^2 f(\bar{\mathbf{x}}) \prec 0$. Hence, the point $\bar{\mathbf{x}}$ is a local maximum.
- $f(\mathbf{y}) \gtrless f(\bar{\mathbf{x}})$ when $\nabla^2 f(\bar{\mathbf{x}}) = 0$. Hence, the point $\bar{\mathbf{x}}$ can be a saddle point (i.e., $f(x) = x^3$ at $\bar{x} = 0$), a local minima (i.e., $f(x) = x^4$ at $\bar{x} = 0$) or a local maxima (i.e., $f(x) = -x^4$ at $\bar{x} = 0$).





Convexity

Definition

A function $f : \mathcal{Q} \to \mathbb{R} \cup \{+\infty\}$ is called convex on its domain $\mathcal{Q}$ if, for any $\mathbf{x}_1,\ \mathbf{x}_2 \in \mathcal{Q}$ and $\alpha \in [0, 1]$, we have:

$$f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \leq \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2).$$

- If $-f(\mathbf{x})$ is convex, then $f(\mathbf{x})$ is called concave.

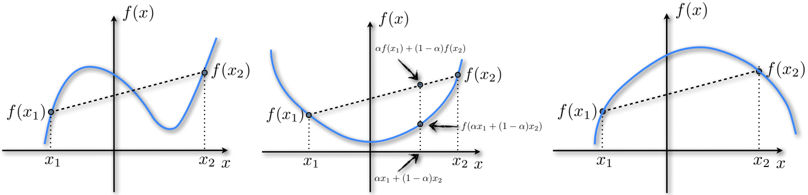


Figure: (**Left**) Non-convex (**Middle**) Convex (**Right**) Concave

Convexity

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$$f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \leq \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2).$$

- Additional terms that you will encounter in the literature

Definition (Proper)

A convex function f is called proper if its domain satisfies $\mathrm{dom}(f) \neq \emptyset$ and, $f(\mathbf{x}) > -\infty,\ \forall x \in \mathrm{dom}(f)$.

Definition (Extended real-valued convex functions)

We define the extended real-valued convex functions f as

$$f(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \in \mathrm{dom}(f) \\ +\infty & \text{if otherwise} \end{cases}$$

To denote this concept, we use $f : \mathrm{dom}(f) \to \mathbb{R} \cup \{+\infty\}$. (Note how l.s.c. might be useful)

Convexity

Definition

A function $f : \mathcal{Q} \to \mathbb{R} \cup \{+\infty\}$ is called convex on its domain $\mathcal{Q}$ if, for any $\mathbf{x}_1,\ \mathbf{x}_2 \in \mathcal{Q}$ and $\alpha \in [0, 1]$, we have:

$$f(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \leq \alpha f(\mathbf{x}_1) + (1-\alpha)f(\mathbf{x}_2).$$

Example

Function	Example	Attributes
ℓ_p vector norms, $p \geq 1$	$\|\mathbf{x}\|_2, \|\mathbf{x}\|_1, \|\mathbf{x}\|_\infty$	convex
ℓ_p matrix norms, $p \geq 1$	$\|\mathbf{X}\|_* = \sum_{i=1}^{\text{rank}(\mathbf{X})} \sigma_i$	convex
Square root function	$\sqrt{x}$	concave, nondecreasing
Maximum of functions	$\max\{x_1, \ldots, x_n\}$	convex, nondecreasing
Minimum of functions	$\min\{x_1, \ldots, x_n\}$	concave, nondecreasing
Sum of convex functions	$\sum_{i=1}^n f_i$, f_i convex	convex
Logarithmic functions	$\log(\det(\mathbf{X}))$	concave, assumes $\mathbf{X} \succ 0$
Affine/linear functions	$\sum_{i=1}^n X_{ii}$	both convex and concave
Eigenvalue functions	$\lambda_{max}(\mathbf{X})$	convex, assumes $\mathbf{X} = \mathbf{X}^T$

Strict convexity

Definition

A function $f : \mathcal{Q} \to \mathbb{R} \cup \{+\infty\}$ is called *strictly convex* on its domain $\mathcal{Q}$ if and only if for any $\mathbf{x}_1, \ \mathbf{x}_2 \in \mathcal{Q}$ and $\alpha \in [0, 1]$ we have:

$$f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) < \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2).$$

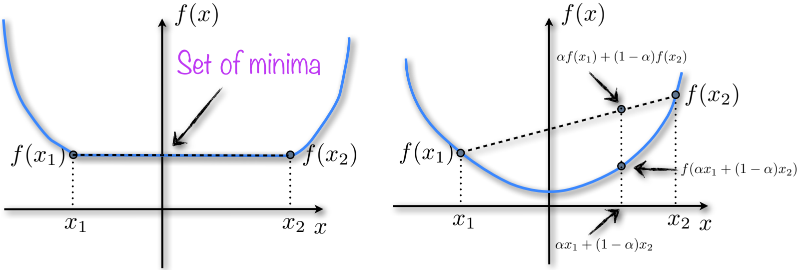


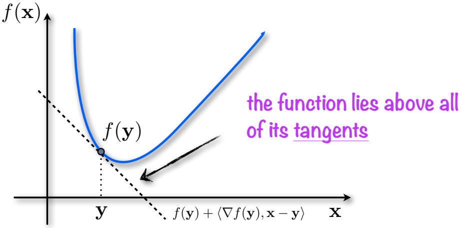
Figure: **(Left panel)** Convex function. **(Right panel)** Strictly convex function.

Revisiting: Alternative definitions of function convexity II

Definition

A function $f \in \mathcal{C}^1(\mathcal{Q})$ is called convex on its domain if for any $\mathbf{x},\ \mathbf{y} \in \mathcal{Q}$:

$$f(\mathbf{x}) \geq f(\mathbf{y}) + \langle \nabla f(\mathbf{y}),\ \mathbf{x} - \mathbf{y} \rangle.$$



Definition

A function $f \in \mathcal{C}^1(\mathcal{Q})$ is called convex on its domain if for any $\mathbf{x},\ \mathbf{y} \in \mathcal{Q}$:

$$\langle \nabla f(\mathbf{y}) - \nabla f(\mathbf{x}),\ \mathbf{y} - \mathbf{x} \rangle \geq 0.$$

*That is, if its gradient is a monotone operator.

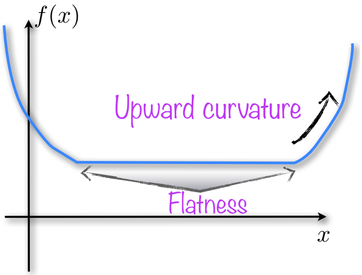
Revisiting: Alternative definitions of function convexity III

Definition

A function $f \in \mathcal{C}^2(\mathbb{R}^p)$ is called convex on its domain if for any $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^p$:

$$\nabla^2 f(\mathbf{x}) \succeq 0.$$

- Geometrical interpretation: the graph of f has zero or positive (upward) curvature.
- However, this does not exclude flatness of f.
- $\nabla^2 f(\mathbf{x}) \succ 0$ is a sufficient condition for *strict* convexity.



Stationary points and convexity

Lemma

Let f be a **smooth convex** *function, i.e., $f \in \mathcal{F}^1$. Then, any stationary point of f is also a global minimum.*

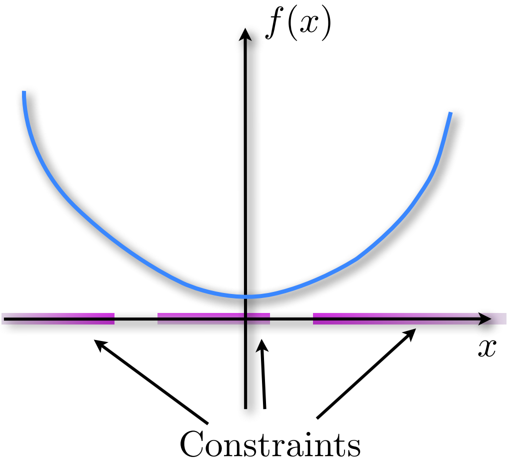
Proof.

Let $\mathbf{x}^\star$ be a stationary point, i.e., $\nabla f(\mathbf{x}^\star) = 0$. By convexity, we have:

$$f(\mathbf{x}) \geq f(\mathbf{x}^\star) + \langle \nabla f(\mathbf{x}^\star), \ \mathbf{x} - \mathbf{x}^\star \rangle \overset{\nabla f(\mathbf{x}^\star)=0}{=} f(\mathbf{x}^\star) \quad \text{for all } \mathbf{x} \in \mathbb{R}^p.$$

□

Is convexity of f enough for an iterative optimization algorithm?



Convexity over sets

Definition

- $\mathcal{Q} \subseteq \mathbb{R}^p$ is a convex set if $\mathbf{x}_1,\ \mathbf{x}_2 \in \mathcal{Q} \Rightarrow \forall \alpha \in [0,1],\quad \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in \mathcal{Q}$.
- $\mathcal{Q} \subseteq \mathbb{R}^p$ is a *strictly* convex set if
$$\mathbf{x}_1,\ \mathbf{x}_2 \in \mathcal{Q} \Longrightarrow \forall \alpha \in (0,1),\quad \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in \text{interior}(\mathcal{Q}).$$

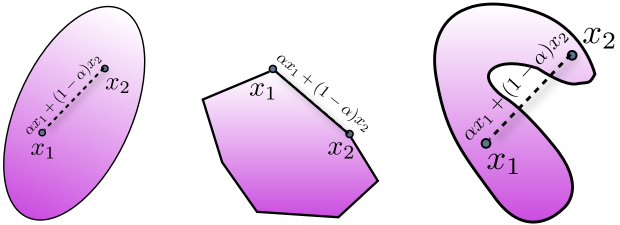


Figure: (**Left**) Strictly convex (**Middle**) Convex (**Right**) Non-convex

Convexity over sets

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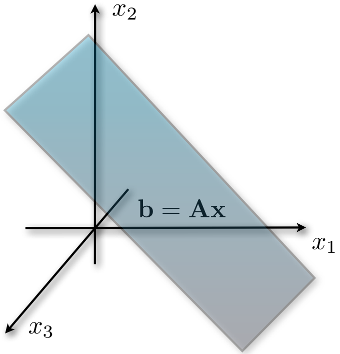

Figure: A linear set of equations $\mathbf{b} = \mathbf{A}\mathbf{x}$ defines an affine (thus convex) set.

Convexity over sets

Definition

- $\mathcal{Q} \subseteq \mathbb{R}^p$ is a convex set if $\mathbf{x}_1,\ \mathbf{x}_2 \in \mathcal{Q} \Rightarrow \forall \alpha \in [0,1],\quad \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in \mathcal{Q}$.
- $\mathcal{Q} \subseteq \mathbb{R}^p$ is a *strictly* convex set if

$$\mathbf{x}_1,\ \mathbf{x}_2 \in \mathcal{Q} \Longrightarrow \forall \alpha \in (0,1),\quad \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in \text{interior}(\mathcal{Q}).$$

Why is this also important/useful?

- convex sets $<>$ convex optimization constraints

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} && f_0(\mathbf{x}) \\ &\text{subject to} && \text{constraints} \end{aligned}$$

Some basic notions on sets I

Definition (Closed set)

A set is called *closed* if it contains all its limit points.

Definition (Closure of a set)

Let $\mathcal{Q} \subseteq \mathbb{R}^p$ be a given open set, i.e., the limit points on the boundaries of $\mathcal{Q}$ do not belong into $\mathcal{Q}$. Then, the closure of $\mathcal{Q}$, denoted as $\mathrm{cl}(\mathcal{Q})$, is the smallest set in $\mathbb{R}^p$ that includes $\mathcal{Q}$ with its boundary points.

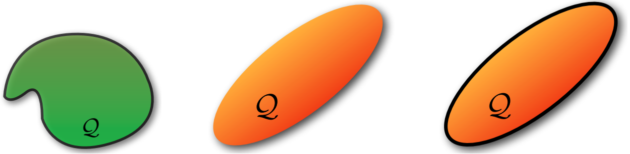

Figure: (**Left panel**) Closed set $\mathcal{Q}$. (**Middle panel**) Open set $\mathcal{Q}$ and its closure $\mathrm{cl}(\mathcal{Q})$ (**Right panel**).

Some basic notions on sets II

Definition (Interior)

Let $\mathcal{Q} \subseteq \mathbb{R}^p$. Then, a point $\mathbf{x} \in \mathbb{R}^p$ is an *interior* of $\mathcal{Q}$ if a neighborhood with radius r of $\mathbf{x}$ is also included in $\mathcal{Q}$. That is, there exists $r > 0$, such that $\{\mathbf{y} \ : \ \|\mathbf{y} - \mathbf{x}\|_2 \leq r\} \in \mathcal{Q}$. The set of all interior points is denoted as int$(\mathcal{Q})$.

Example

- The interior of an open set is the set itself.
- The interior of the set $\{\mathbf{x} \ : \ \|\mathbf{y} - \mathbf{x}\|_2 \leq r\}$ is the open set $\{\mathbf{x} \ : \ \|\mathbf{y} - \mathbf{x}\|_2 < r\}$.

Some basic notions on sets II

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Example

- The interior of an open set is the set itself.
- The interior of the set $\{\mathbf{x} \ : \ \|\mathbf{y} - \mathbf{x}\|_2 \leq r\}$ is the open set $\{\mathbf{x} \ : \ \|\mathbf{y} - \mathbf{x}\|_2 < r\}$.

Definition (Relative interior)

Let $\mathcal{Q} \subseteq \mathbb{R}^p$. Then, a point $\mathbf{x} \in \mathbb{R}^p$ is a *relative interior* of $\mathcal{Q}$ if $\mathcal{Q}$ contains the intersection of a neighborhood with radius r around $\mathbf{x}$ with the intersection of all affine sets containing $\mathcal{Q}$, i.e., $\text{aff}(\mathcal{Q})$. The set of all relative interior points is denoted as $\text{relint}(\mathcal{Q})$.

Example

The interior of the affine set $\mathcal{X} = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$ is empty. However, its relative interior is itself, i.e., $\text{relint}(\mathcal{X}) = \mathcal{X}$.

Convex hull

Definition (Convex hull)

Let $\mathcal{V} \subseteq \mathbb{R}^p$ be a set. The convex hull of $\mathcal{V}$, i.e., $\mathrm{conv}(\mathcal{V})$, is the *smallest* convex set that contains $\mathcal{V}$.

Definition (Convex hull of points)

Let $\mathcal{V} \subseteq \mathbb{R}^p$ be a finite set of points with cardinality $|\mathcal{V}|$. The convex hull of $\mathcal{V}$ is the set of all convex combinations of its points, i.e.,

$$\mathrm{conv}(\mathcal{V}) = \left\{ \sum_{i=1}^{|\mathcal{V}|} \alpha_i \mathbf{x}_i \ : \ \sum_{i=1}^{|\mathcal{V}|} \alpha_i = 1,\ \alpha_i \geq 0, \forall i,\ \mathbf{x}_i \in \mathcal{V} \right\}.$$

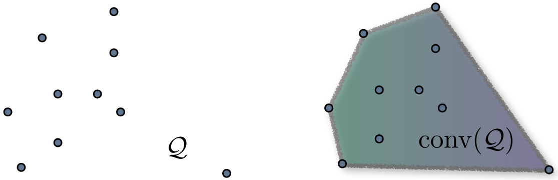


Figure: (**Left**) Discrete set of points $\mathcal{V}$. (**Right**) Convex hull $\mathrm{conv}(\mathcal{V})$.

Revisiting: Alternative definitions of function convexity IV

Definition

The epigraph of a function $f : \mathcal{Q} \to \mathbb{R}, \mathcal{Q} \subseteq \mathbb{R}^p$ is the subset of $\mathbb{R}^{p+1}$ given by:

$$\text{epi}(f) = \{(\mathbf{x}, w) : \mathbf{x} \in \mathcal{Q}, w \in \mathbb{R}, f(\mathbf{x}) \leq w\}.$$

Lemma

A function $f : \mathcal{Q} \to \mathbb{R}$ is convex if and only if its epigraph, i.e, the region above its graph, is a convex set.

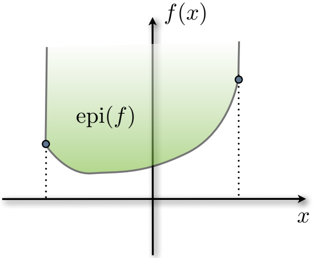


Figure: Epigraph — the region in green above graph $f(\cdot)$.

Unfortunately, convexity does not imply tractability

But first...

1. How do we define tractability?
2. How do we classify running times of algorithms?

Asymptotic Notation

What is this notation?

- Asymptotic Notation (Big-Oh Notation, Landau's notation) describes asymptotic growth of functions.
- It is usually used to describe:
  - Running time of an algorithm
  - Memory storage require by an algorithm
  - Error achieved by an approximation
- Exact computation of the running time, memory, or error is usually not important: For large inputs, multiplicative constants and lower-order terms "do not matter."

Examples

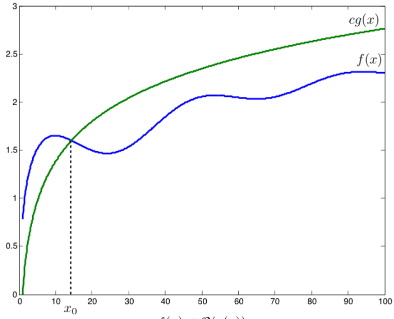
- Binary search's running time in a sorted list of n elements. [1]: $O(\log(n))$
- Number of comparisons required for sorting a list of n elements [1]: $\Omega(n \log(n))$

Asymptotic Notation: Big-Oh

Definition (Big-Oh)

Let f, g be two functions defined on some subset of the real numbers:

$$f(x) \in O(g(x)) \text{ iff } \exists c > 0, \exists x_0, \text{such that } |f(x)| \leq c|g(x)|, \forall x \geq x_0$$



$f(x) \in \mathcal{O}(g(x))$

- In computer science, the definition is taken over positive integers.

Example

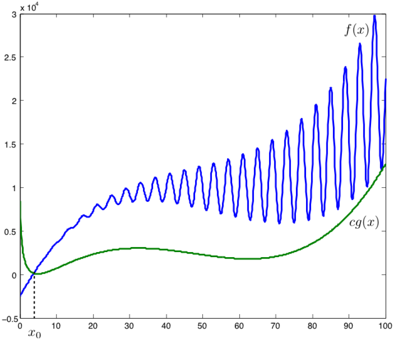
- $x \in O(x^2)$
- $\log(n!) \in O(n \log(n))$ [cf., lab 1]
- $n^{1+\sin(n)} \in O(n^2)$

Asymptotic Notation: Big-Omega

Definition (Big-Omega)

Let f, g be two functions defined on some subset of the real numbers:

$$f(x) \in \Omega(g(x)) \text{ iff } \exists c > 0, \exists x_0, \text{such that } |f(x)| \geq c|g(x)|, \forall x \geq x_0$$



$f(x) \in \Omega(g(x))$

- Intuition: g is a lower bound of f iff f is an upper bound of g.
- $f(x) \in \Omega(g(x)) \Leftrightarrow g(x) \in O(f(x))$.

Example

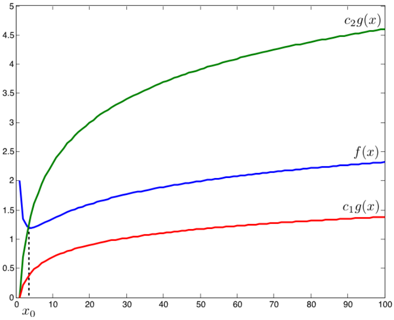
- $x^2 \in \Omega(x)$
- $\log(n!) \in \Omega(n \log(n))$ [cf., lab 1]
- $n^{1+\sin(n)} \in \Omega(1)$

Asymptotic Notation: Theta

Definition (Theta)

Let f, g be two functions defined on some subset of the real numbers:

$$f(x) \in \Theta(g(x)) \text{ iff } \exists c_1, c_2 > 0, \exists x_0, \text{such that } c_1 \leq \frac{|f(x)|}{|g(x)|} \leq c_2, \forall x \geq x_0$$



$f(x) \in \Theta(g(x))$

- Intuition: g is a tight bound for f iff it is both an upper and a lower bound of it.
- $f(x) \in \Theta(g(x))$ iff $f(x) \in O(g(x))$ and $f(x) \in \Omega(g(x))$.
- $f(x) \in \Theta(g(x))$ iff $g(x) \in \Theta(f(x))$.

Example

- $\sin(x) \in \Theta(1)$
- $x + \log(x) \in \Theta(x)$
- $\log(n!) \in \Theta(n \log(n))$ [cf., lab 1]
- **Stirling's approximation:** $n! \in \sqrt{2\pi n}(\frac{n}{e})^n(1 + \Theta(\frac{1}{n}))$

Asymptotic Notation: small-oh and small-omega

Definition (small-oh, small-omega)

Let f, g be two functions defined on some subset of the real numbers:

$$f(x) \in o(g(x)) \text{ iff } \forall c > 0, \exists x_0, \text{such that } |f(x)| \leq c|g(x)|, \forall x \geq x_0,$$

or equivalently $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = 0$.

$$f(x) \in \omega(g(x)) \text{ iff } \forall c > 0, \exists x_0, \text{such that } |f(x)| \geq c|g(x)|, \forall x \geq x_0,$$

or equivalently $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} = \infty$.

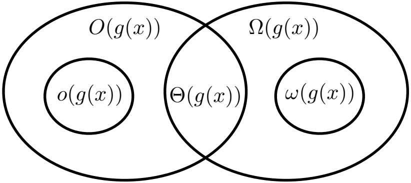
- These are non-tight upper/lower bounds.
- $g(x) \in o(f(x))$: g is dominated by f asymptotically.
- $f(x) \in \omega(g(x))$: f dominates g asymptotically.
- $f(x) \in \omega(g(x)) \Leftrightarrow g(x) \in o(f(x))$.

Example

- $\frac{1}{x} \in o(1)$
- $5 \in \omega(\frac{1}{x})$
- $n! \in o(n^n)$ [cf., lab 1]
- $n! \in \omega(2^n)$ [cf., lab 1]

Hierarchy of asymptotic notation classes

- Relation between the different asymptotic notations:



- Analogy with real numbers comparison:

Aymptotic function comparison	Real numbers comparison
$f(x) = O(g(x))$	$a \leq b$
$f(x) = \Omega(g(x))$	$a \geq b$
$f(x) = \Theta(g(x))$	$a = b$
$f(x) = o(g(x))$	$a < b$
$f(x) = \omega(g(x))$	$a > b$

- Difference from real numbers comparison: Not all functions are **asymptotically comparable**, e.g., n, $n^{1+\sin(n)}$.

Asymptotic Notation: some remarks

Some notation abuse:

- Use of equality: $f(x) = O(g(x))$

Some variations:

- Soft-Oh: $\tilde{O}(\cdot)$ notation ignores log terms, i.e., $O(x^c \log^k(x)) = \tilde{O}(x^c)$.
- Asymptotic notation can also describe limiting behavior as $x \to a$, e.g., $e^x = 1 + x + \frac{x^2}{2} + o(x^2), x \to 0$ (by Taylor's theorem).

Computational complexity: Complexity of deciding

Decision problems: “yes” or “no” answers.

How hard are these problems?

- Shortest path: Is there a path from point a to point b shorter than d ?
- Subset sum problem: Is there a subset of some given integers that sums up to d ?

For $d = 5$?

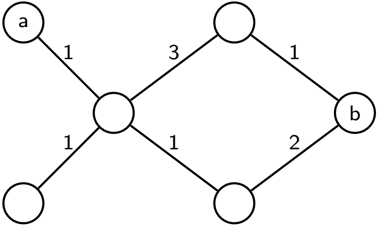


Figure: Graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$.

Applications:

- Driving directions in google maps.
- Minimum delay path for data packets in networking.

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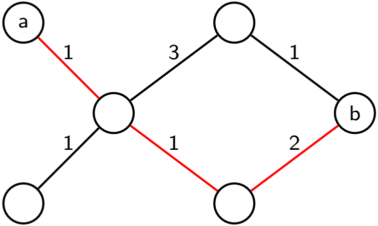
For $d = 5$? yes.

Shortest path can be computed by Dijkstra in $O(|\mathcal{V}|^2)$ [1]



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How hard are these problems?

- Shortest path: Is there a path from point a to point b shorter than d ?
- Subset sum problem: Is there a subset of some given integers that sums up to d ?

For $d = 0$?

Consider these two lists of integers:

$$A : -2, 5, 4, 9, 19, -11$$

$$B : -2, 5, 4, 9, 19, -6$$

Applications:

- In cryptography: public key system, computer passwords, message verification.

Computational complexity: Complexity of deciding

Decision problems: "yes" or "no" answers.

How hard are these problems?

- Shortest path: Is there a path from point a to point b shorter than d ?
- Subset sum problem: Is there a subset of some given integers that sums up to d ?

For $d = 0$?

Consider these two lists of integers:

$$A : -\mathbf{2}, 5, \mathbf{4}, \mathbf{9}, 19, -\mathbf{11} \quad \boxed{\text{Yes}}$$

$$B : -2, 5, 4, 9, 19, -6 \quad \boxed{\text{No}}$$

Applications:

- In cryptography: public key system, computer passwords, message verification.

Computational complexity: Complexity of deciding

Decision problems: “yes” or “no” answers.

How hard are these problems?

- Shortest path: Is there a path from point a to point b shorter than d ?
- Subset sum problem: Is there a subset of some given integers that sums up to d ?

No known “efficient” algorithm can decide if a general list has a d-subset sum

-84	348	422	818	-364	44	-843	222	978	-934	452	83	-819	-100	474	179	-134
-951	515	-790	779	912	291	-560	585	-30	-255	693	-803	-932	-675	668	746	-323
46	-636	918	980	-943	22	-30	676	-766	-287	108	715	-352	530	673	-92	-91
979	239	-153	-353	-206	324	31	375	717	-451	685	-915	-668	967	-934	968	540
447	540	120	975	-445	-538	678	-838	-316	-471	-903	-716	997	-714	825	-480	512
-206	-158	-803	228	-276	966	-174	613	-367	489	518	-565	-169	656	-811	382	985
182	-122	-34	304	561	-829	-196	-731	-245	-737	352	555	509	763	702	194	-594
146	-22	-693	203	-109	-175	906	140	740	606	749	-285	387	-860	44	-442	-249
15	-567	-383	596	-886	971	-330	149	63	-391	-338	-950	542	-44	-968	-442	418
-709	666	-739	221	-904	-29	-733	-968	-950	-314	37	668	825	478	-742	368	-673
-418	3	-313	-246	62	224	391	870	-504	319	-92	-274	-379	468	186	623	-352
2	-795	326	-651	534	-978	846	230	448	-923	-641	577	-591	633	444	-848	618
-717	-271	32	-881	229	537	-25	337	-135	545	695	760	-855	67	-926	-261	-562
-187	642	594	-696	865	483	11	-157	-477	380	-908	353	174	122	-648	613	-343
186	-755	153	-233	563	-566	-991	406	-621	855	47	91	85	-307	-535	917	-76
619	299	-867	-974	510	-205	-623	-59	-571	-876	-535	798	288	102	430	988	-295
-59	473	-503	931	-348	-475	560	-182	-10	375	807	-801	196	-958	541	-149	-853

Applications:

- In cryptography: public key system, computer passwords, message verification.

Computational complexity: Class $\mathcal{P}$

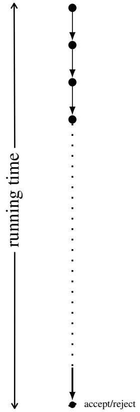


Figure: Deterministic Turing machine

Definition (Class $\mathcal{P}$)

$\mathcal{P}$ (polynomial time): decision problems solvable in polynomial time.

Definition (Turing machine)

(Deterministic) Turing machine (DTM): mathematical computational model, think of it as your regular computer. For formal definition, refer to [4].

- Problems in $\mathcal{P}$ can be solved by a DTM in polynomial time.
- Polynomial time means $O(n^k)$ time for some $k \in \mathbb{N}$, where n is the size of the input.

Example (Shortest path problem)

Shortest path can be computed by Dijkstra in $O(|\mathcal{V}|^2)$ [1]

Computational complexity: Class $\mathcal{NP}$

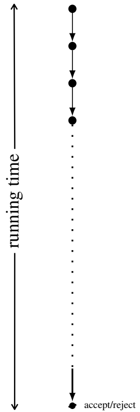


Figure: Deterministic Turing machine

Definition (Class $\mathcal{NP}$)

$\mathcal{NP}$ (nondeterministic polynomial time): decision problems such that "yes" answer can be "checked" in poly-time via a deterministic Turing machine, i.e., there exists a certificate (proof) whose correctness can be verified in poly-time.

- Polynomial time means $O(n^k)$ time for some $k \in \mathbb{N}$, where n is the size of the input.
- It follows then that the certificate should be of polynomial length.
- $\mathcal{P} \subseteq \mathcal{NP}$

Example

- Subset sum problem:
  - Proof: Subset of integers that do sum up to d.
  - Verification: Addition of can be done in polynomial time.
- Shortest path problem.

Computational complexity: Class $\mathcal{NP}$

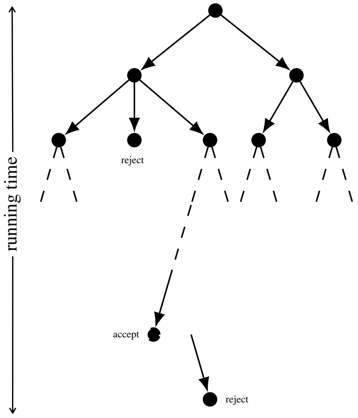


Figure: Non-Deterministic (decider) Turing machine

Definition (Class $\mathcal{NP}$)

$\mathcal{NP}$ (nondeterministic polynomial time): decision problems such that "yes" answer can be "checked" in poly-time via a deterministic Turing machine, i.e., there exists a certificate (proof) whose correctness can be verified in poly-time.

Definition (Non-deterministic TM)

Non-deterministic Turing machine (NTM): A fictional "super" computer than can "clone" itself every time it reaches a decision, each clone continue with one of the possible choices. For formal definition, refer to [4].

- Problems in $\mathcal{NP}$ can be solved by a NTM in polynomial time.

Computational complexity: Open Problem

Open problem: $\mathcal{P}$ vs $\mathcal{NP}$

- $\mathcal{P} \stackrel{?}{=} \mathcal{NP}$: Is generating a proof as easy as checking it ?
- One of the 7 Millennium Prize Problems by the Clay Mathematics Institute.
- Conjecture: $\mathcal{P} \neq \mathcal{NP}$.
- Many other open problems in complexity.

Computational complexity: Reductions

Definition (Polynomial time reducibility)

- Reduction is a way of converting one problem to another such that the solution to the second problem can be used to solved the first problem.
- Polynomial time reduction (mapping reducibility) [4]: We say a problem A is poly-time mapping reducible to B, denoted by $A \leq_p B$, iff there exists polynomial time computable function f that converts any input of A into an input of B, such that the transformed problem has the same output as the original problem.
- A's answer is yes on input $w \Leftrightarrow B$'s answer is yes on input $f(w)$.

- If A is poly-time reducible to B, then the existence of a poynomial algorithm of B would imply the existence of a polynomial algorithm for A also.

Computational complexity: Hardness

Definition ($\mathcal{NP}$-Hard)

- $\mathcal{NP}$-Hard: Problems (not necessarily decision) that are at least as hard as the hardest problems in $\mathcal{NP}$.
- $B \in \mathcal{NP}$-Hard $\Leftrightarrow \forall A \in \mathcal{NP}, A \leq_p B$.
- Examples: search version of subset sum, candy crush.

Definition ($\mathcal{NP}$-Complete)

- $\mathcal{NP}$-Complete: Decision problems in $\mathcal{NP}$, that are at least as hard as the hardest problems in $\mathcal{NP}$.
- $\mathcal{NP}$-Complete $= \mathcal{NP} \cap \mathcal{NP}$-Hard.
- Subset sum, Karp's 21 NP-complete problems [2].

Computational complexity: Class $co\mathcal{NP}$

Definition (Class $co\mathcal{NP}$)

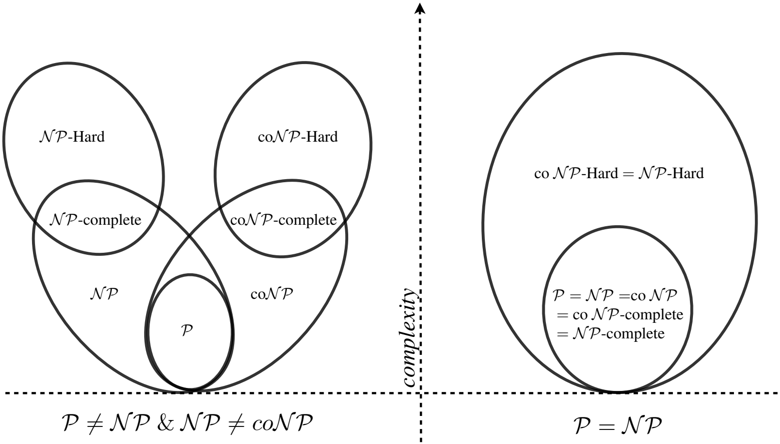
$co\mathcal{NP}$ (complement nondeterministic polynomial time): decision problems such that "no" answer can be "checked" in poly-time, i.e., there exists a certificate (or a counter-example) whose correctness can be verified in poly-time.

Example ($\overline{\text{subset sum}} \in co\mathcal{NP}$)

Does every non-empty subset sums up to a non-zero sum?

- $A \in co\mathcal{NP}$ iff $\overline{A} \in \mathcal{NP}$
- $\mathcal{P} \in co\mathcal{NP} \cap \mathcal{NP}$
- $B \in co\mathcal{NP}$-Hard $\Leftrightarrow \forall A \in co\mathcal{NP}, A \leq_p B$
- co$\mathcal{NP}$-Complete: $co\mathcal{NP} \cap co\mathcal{NP}$-Hard.

Two of the possible worlds



Hardness result: Certifying optimality in mathematical programming

Problem (Smooth constrained optimization problems)

We consider smooth constrained optimization problems:

$$\min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x}) \textit{ such that } g_i(\mathbf{x}) \leq 0, \forall i \in [1, \cdots, m]$$

Smooth: we assume that f and all g_i's are infinitely differentiable.

How hard is it to check that a given solution $\mathbf{x} \in \mathbb{R}^p$ is optimal?

Why should we care ?

- Optimization is ubiquitous: applications in control, estimation, signal processing, electronics design, communications, finance, ...
- Emphasize the importance of convexity: smoothness alone is not enough.

Hardness result: Certifying optimality in mathematical programming

Problem (Smooth constrained optimization problems)

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How hard is it to check that a given solution $\mathbf{x} \in \mathbb{R}^p$ is optimal?

Example (Subset sum problem revisited)

Subset Sum problem: Given integers $d_1 \cdots d_p$, is there a non-empty subset that sums up to d_0?

Equivalent smooth constrained optimization problem:

$$\min_{\mathbf{y}\in\mathbb{R}^p} \left(\sum_{i=1}^p d_i y_i - d_0\right)^2 + \sum_{i=1}^p y_i(1-y_i) \text{ such that } 0 \leq y_i \leq 1, \forall i \in [1, \cdots, p]$$

Hardness result: certifying optimality in mathematical programming

Proposition

Checking if a point is the global minimum of a smooth constrained optimization problem is NP-Hard [3] in general.

Proof.

Reduce an NP-complete problem to our problem:

- B: "Compute the global minimum of a smooth constrained optimization problem".
- Subset sum problem (A) is known to be NP-complete.
- It has an equivalent formulation:

$$\min_{\mathbf{y}\in\mathbb{R}^p} \left(\sum_{i=1}^{p} d_i y_i - d_0\right)^2 + \sum_{i=1}^{p} y_i(1-y_i) \text{ such that } 0 \leq y_i \leq 1, \forall i \in [1,\cdots,p] \quad (1)$$

- Zero is the global minimum objective value of (1) iff there exists a subset of $[d_1,\cdots,d_p]$ with sum d_0.
- We showed that $A \leq_p B$, and $A \in \mathcal{NP}$-complete, then $B \in \mathcal{NP}$-Hard.

□

Hardness result: Certifying optimality in mathematical programming

Proposition

Checking if a point is a local minimum of a smooth constrained optimization problem is coNP-Hard [3] in general.

We need a structure beyond smoothness that avoids such problems: Convexity?

Unfortunately, convexity does not imply tractability

Consider the following NP-Hard problem:

Problem (Maximum Cut)

Given a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, such that $n = |\mathcal{V}|, m = |\mathcal{E}|$, the maximum cut problem is the problem of finding a cut (i.e., a partition of the vertices of a graph into two disjoint subsets $\mathcal{S}$ and $\bar{\mathcal{S}}$) with a cut-set (edges between $\mathcal{S}$ and $\bar{\mathcal{S}}$) of maximum weight.

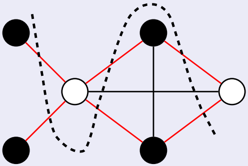

Figure: The set $\mathcal{S}$ of black nodes corresponds to the cut-set $\delta(\mathcal{S})$ of red edges.

Max-Cut problem can be formulated as: $\max_{\mathcal{S} \subseteq \mathcal{V}} \mathbf{w}^T \delta(\mathcal{S})$, *where* $\mathbf{w} \in \mathbb{R}^m$ *denote the edge weights.*

Unfortunately, convexity does not imply tractability

Example (Cut polytope)

Consider the following smooth convex constrained optimization problem:

$$\max_{\mathbf{x} \subseteq \mathsf{Cut}_n} \mathbf{w}^T \mathbf{x} \tag{2}$$

where Cut_n is the convex hull of the characteristic vectors of cut sets, i.e., $\mathsf{Cut}_n = \mathrm{conv}(\{\mathbb{1}_{\mathcal{S}}, \mathcal{S} \in \mathcal{V}\})$. It is called the cut polytope. Problem (2) is NP-Hard, since Max-Cut problem can be reformulated as (2).

Convexity is still helpful

- Convexity does not imply tractability in general.
- Convexity implies that finding a local minimum is enough to find a global minimum.

References I

[1] T.H. Cormen, C.E. Leiserson, R.L. Rivest, and C. Stein.
Introduction To Algorithms.
MIT Press, 2001.

[2] Richard M Karp.
Reducibility among combinatorial problems.
Springer, 1972.

[3] Katta G Murty and Santosh N Kabadi.
Some np-complete problems in quadratic and nonlinear programming.
Mathematical programming, 39(2):117–129, 1987.

[4] Michael Sipser.
Introduction to the Theory of Computation.
Cengage Learning, 2012.