Homework 1

Assigned: 26/09/2011.
Due: 07/10/2011.

Exercise 1. DNA FINGERPRINTING EVIDENCE
There's been a gruesome crime, but DNA samples were collected. DNA fingerprinting is not perfect: the probability for a match, given the defendant is innocent, is $1/100000$. If the defendant is guilty, a match always occurs. The defendant is part of a group of $N \geq 1$ people, which could have committed the crime just as well (in the sense, that there is no other evidence). Given there is a match (to the defendant's DNA), how should the court of justice conclude?

1. **[2 points]** Setup the problem: Define relevant variables, along with their value range. Write down the joint distribution as a table. Write down the conditional distribution of a DNA match occuring or not, conditioned on the other relevant variable (as a conditional probability table). Assume that each of the N people could have committed the crime equally likely (assessed before any DNA samples had been found), and that no other evidence whatsoever is available.

2. **[4 points]** State the probability of a match (this will be a function of N). A member of the jury (not trained in Bayesian statistics): "We *know* there is a match! So what's the point of computing that belief?". Explain him/her (1-2 sentences).
   State the probability of the defendant being guilty, given the match occured. Plot it as a function of N. For which N is the probability equal to $1/2$?
   The traditional (but wrong) way to look at problems like this is to look at likelihood ratios: probability of match given defendant guilty, divided by probability of match given defendant innocent. Explain why this is dangerous (or unfair, to say the least).

Exercise 2. SEATS IN AIRPLANE
[5 points] N people board an airplane with $N > 1$ seats. They've been assigned seat numbers, but the first person is sleepy and picks a seat at random. Subsequent passengers are careful and choose their assigned seat if free, but if taken, they pick another free seat at random. What is the probability that the last passenger to board sits in his/her assigned seat?
Hints: You may assume that people have been assigned numbers $1, \ldots, N$, and also board in that order (think for a second why). Try an inductive argument. Let P_N be the probability that the last person gets his/her assigned seat. What is P_2? For $N > 2$, what happens if the first person picks seat 1? Seat N? If he/she picks seat $1 < i < N$, the ith person will be in a situation that you should compare to what a *first* person would do in a smaller plane (argument!).

Exercise 3. THE MARTINGALE
Here's a great idea for riskless profit. Somebody offers you a fair game. A fair coin is tossed. Tails: you lose your bet. Heads: you keep your bet, and get paid the same amount (so you end up doubling your bet).

1. **[2 points]** The Martingale amounts to the following strategy. You start with bet 1. Whenever you lose, you double your bet. Let N be the numbers of games you play until you win. Compute the probability distribution for N. What is the probability that $N < \infty$? What is your gain (at the point you win)?

2. **[2 points]** Let L be the amount of money you lost just before you win (at game number N). What is the expected value of L? What does that mean for you? For the person/casino you play against?

3. **[3 points]** Suppose you have a budget of size $B = 2^k$. You play the Martingale, but when you run out of money, you have to stop (being bankrupt). What is your expected gain in this case? What you will find, is a special instance of the optional stopping theorem.
   Hint: What is the probability of $N > k$? You might want to use the identity

$$\sum_{j=0}^{n-1} q^j = \frac{q^n - 1}{q - 1}, \quad q \neq 1.$$

   And think about how much you really lose when you run against your limit.